

# COMPUTING HIGHER INDICATORS FOR THE DOUBLE OF A SYMMETRIC GROUP

REBECCA COURTER

**ABSTRACT.** In this paper we explicitly determine all indicators for the Drinfel'd doubles of the symmetric group acting upon up to 10 objects. We explore when distinct characters give exactly the same indicators and when the indicators have a zero value. We find that the indicators are all non-negative integers, which supports our conjecture that the indicators for the Drinfel'd double of any symmetric group will be non-negative integers, just as they are for the symmetric groups themselves.

## 1. INTRODUCTION

For an irreducible representation of a group over the complex numbers, the classical Frobenius-Schur indicator determines whether or not the representation is defined over the reals. One may also define higher indicators for representations of a group, so that the classical one is the second indicator. The classical Frobenius-Schur indicator extends to any semisimple Hopf algebra [LM], and the higher indicators also extend to Hopf algebras [KSZ2].

Indicators, including their higher analogues, are invariants that are proving very useful in the study of Hopf algebras. They have been used in classification problems [K] [NS1]; in studying possible dimensions of the representations of a semisimple Hopf algebra [KSZ1]; and in determining the prime divisors of the exponent of a Hopf algebra [KSZ2] [NS2]. Moreover, the indicator is invariant under equivalence of monoidal categories [MaN]. Another area where indicators are proving useful is conformal field theory; see the work of Bantay [B1] [B2]. The notion of higher indicators has also been extended to more general categories [NS1] [NS2] [NS3], where quasi-Hopf algebras play a unifying role [N1] [N2].

Frobenius and Schur gave a formula for the classic indicator  $\nu$  of a representation of a finite group  $G$  [Se]. They showed that for any irreducible representation,  $\nu$  was always 1, 0, or  $-1$ , and  $\nu = 1$  occurred precisely when the representation could be defined over  $\mathbb{R}$ . Similarly [LM] gave a version of this formula for a Hopf algebra  $H$ , and again for  $V$  a simple  $H$ -module, with character  $\chi$  and indicator  $\nu(V) = \nu(\chi)$ , the only possible values of  $\nu(V)$  are 0, 1, and  $-1$ . A little earlier the formula was extended for the special case of Kac algebras over  $\mathbb{C}$  in [FGSV]. The symmetric group  $S_n$  has been an object of interest for years. In the 1940's Young showed that the indicator for any irreducible representation of any symmetric group  $S_n$  is 1. This fact was later extended to the Hopf algebra  $D(S_n)$  in [KMM].

The higher indicators of modules of finite groups are denoted  $\nu_m$  for any  $m \in \mathbb{Z}$ , in which the classical Frobenius-Schur indicator  $\nu$  is equal to  $\nu_2$ . [KSZ2] extended the higher indicator formula to Hopf algebras. It is well known that the higher indicators for modules over groups are integers, but for Hopf algebras in general the values may involve roots of unity. However, for  $G$  a finite group, its Drinfel'd double  $D(G)$  is a

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nice Hopf algebra, with a braiding on its module category, so one should expect it to behave better.

In his dissertation [Ke], Keilberg examined the Drinfel'd double of any dihedral group, semi-dihedral group, and some other classes of non-abelian groups of order  $pq$ . He proved that all indicators of these Hopf algebras are integers by explicitly finding their values. He also proved that for the Drinfel'd double of the diheadral group, the indicators are also non-negative. As seen before with the classical indicator, the symmetric group is also a very nice object to work with. Scharf proved that all higher indicators of a simple  $S_n$ -module are *non-negative* integers [S]. Whether or not this stronger statement of non-negativity is true for a simple module of  $D(S_n)$ , was the motivating question for this work. This thesis gives positive supporting evidence for this conjecture.

All the indicators of  $D(S_n)$  in this thesis were calculated using GAP, which stands for "Groups, Algorithms, Programming." GAP is a mathematical software system for computational discrete algebra, with particular emphasis on computational Group Theory. [GAP]

This dissertation is organized as follows: in Section 2 we first review general definitions and theorems, and describe how we compute the higher indicators. We then consider how automorphisms of  $G$  affect characters and indicators of  $D(G)$ . In Section 3 we find a more computer-efficient formula to calculate the indicators, we show that we need only find a few higher indicators in order to find them all, and we make observations about what our calculations show. We derive lemmas and propositions describing when representation have the same set of indicator values (that is, when they are I-equivalent), why certain indicators are 0, and when the set we use to calculate the indicators is not empty. In Section 4 we present the indicators and specific details about how they were calculated for  $D(S_3)$  through  $D(S_6)$ . In Section 5 we present the GAP functions written and used to do our calculations, as well as a few others used to give additional information. Appendix A contains the character tables used to compute the indicators for  $D(S_3)$  through  $D(S_6)$ . Due to the large number of tables required to display the indicators for  $D(S_7)$  through  $D(S_{10})$ , those are recorded in Appendix B which is not included in this paper: Appendix B can be found at [http://www.pasadena.edu/files/syllabi/recourter\\_31371.pdf](http://www.pasadena.edu/files/syllabi/recourter_31371.pdf). Also due to the large amount of data computed, no details about how the indicators of  $D(S_7)$  through  $D(S_{10})$  were calculated are given (as was done for the smaller doubles in Section 4).

## 2. PRELIMINARIES

### 2.1. Definitions and Notation

Let  $k$  be an algebraically closed field of characterist 0.

**Definition 2.1.1.** A finite dimensional *Hopf algebra* over  $k$  is a sextuple  $(H, m, u, \Delta, \epsilon, S)$  where  $H$  is a  $k$ -vector space,  $(H, m, u)$  is an algebra with associative multiplication  $m : H \otimes H \rightarrow H$  and unit  $u : k \rightarrow H$ ,  $(H, \Delta, \epsilon)$  is a coalgebra with coassociative comultiplication  $\Delta : H \rightarrow H \otimes H$  and counit  $\epsilon : H \rightarrow k$  such that both  $\Delta$  and  $\epsilon$  are algebra maps, and the antipode  $S : H \rightarrow H$  is an inverse for the identity under convolution. We use the sigma notation for comultiplication,

$$\Delta(h) = \sum h_1 \otimes h_2, \text{ for } h \in H.$$

Thus in sigma notation,  $S$  satisfies for each  $h \in H$ ,

$$\sum S(h_1)h_2 = \sum h_1S(h_2) = \epsilon(h)1.$$

For a reference see [Mo].

To define the Drinfel'd double of a group we must first define the pieces that make up the double.

**Definition 2.1.2.** For any group  $G$ , the *group algebra* over the complex numbers is the collection of formal sums

$$\mathbb{C}G = \left\{ \sum \alpha_g g \mid \alpha_g \in \mathbb{C} \right\},$$

where addition is defined by  $\alpha g + \beta g = (\alpha + \beta)g$ , for all  $\alpha, \beta \in \mathbb{C}$ ,  $g \in G$  and extended component wise, and multiplication is defined by  $\alpha g \cdot \beta h = \alpha\beta(gh)$ , for all  $\alpha, \beta \in \mathbb{C}$ ,  $g, h \in G$  and extended component wise.

**Definition 2.1.3.** The *dual space* of  $\mathbb{C}G$ , notated by  $(\mathbb{C}G)^*$  or  $\mathbb{C}^G$ , is the vector space of all  $\mathbb{C}$ -linear maps from  $\mathbb{C}G \rightarrow \mathbb{C}$ . Its basis is the collection of “coordinate functions”  $\{p_g \mid g \in G\}$  where for  $x \in G$ ,

$$p_g(x) = \delta_{g,x} = \begin{cases} 1 & g = x \\ 0 & g \neq x \end{cases}.$$

For  $g, h, x \in G, \alpha \in \mathbb{C}$ , it's addition is defined by  $(p_g + p_h)(x) = p_g(x) + p_h(x)$ , and scalar multiplication defined by  $(\alpha p_g)(x) = p_g(\alpha x) = \alpha \cdot p_g(x)$ .  $\mathbb{C}^G$  is also an algebra with multiplication defined by  $(p_g \cdot p_h)(x) = p_g(x) \cdot p_h(x) = \delta_{g,h} p_g(x)$ .

We remark that both  $\mathbb{C}G$  and  $\mathbb{C}^G$  are coalgebras and Hopf algebras. The coalgebra structure of  $\mathbb{C}G$  is given by  $\Delta(\sum \alpha_g g) = \sum \alpha_g \Delta(g) = \sum \alpha_g (g \otimes g) = \sum \alpha_g g \otimes g$ , and  $\epsilon(\sum \alpha_g g) = \sum \alpha_g \epsilon(g) = \sum \alpha_g$ . The coalgebra structure of  $\mathbb{C}^G$  is given by  $\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v$ , and the counit is  $\epsilon(p_g) = 1$ .

We can further define an action  $\rightharpoonup: G \otimes \mathbb{C}^G \rightarrow \mathbb{C}^G$  of  $G$  on  $\mathbb{C}^G$  by,

$$x \rightharpoonup p_g := p_{xgx^{-1}}.$$

With this action, we can now define the Drinfel'd double of a group.

**Definition 2.1.4.** [Mo] The *Drinfel'd double* of a group  $G$ , denoted by  $D(G) = \mathbb{C}^G \bowtie \mathbb{C}G$  is a semisimple Hopf algebra with basis elements written as  $p_g \bowtie x$ , where  $g, x \in G$  and  $p_g \in \mathbb{C}^G$ . Multiplication is defined by

$$\begin{aligned} (p_k \bowtie z) \cdot (p_h \bowtie y) &:= p_k (z \rightharpoonup p_h) \bowtie zy = p_k p_{zhz^{-1}} \bowtie zy \\ &= \delta_{k, zhz^{-1}} p_k \bowtie zy. \end{aligned}$$

Comultiplication is defined by

$$\Delta(p_g \bowtie x) := \sum_{h \in G} (p_h \bowtie x) \otimes (p_{h^{-1}g} \bowtie x).$$

The counit is given by  $\epsilon(p_g \bowtie x) = \delta_{g,1}$ , and the antipode is given by  $S(p_g \bowtie x) := p_{x^{-1}g^{-1}x} \bowtie x^{-1}$ .

A further equivalent notation to use for  $D(G)$  is the smash product  $\mathbb{C}^G \# \mathbb{C}G$ , where the elements can be written as  $p_g \# x$ , or simply  $p_g x$ . For a reference see [KMM].

Before we can define indicators, we must first discuss representations and characters. For a reference see [Se]. For an arbitrary group  $G$ :

**Definition 2.1.5.** A *representation of  $G$  over  $\mathbb{C}$*  is a group homomorphism  $\rho : G \rightarrow GL_n(\mathbb{C})$ . The degree of  $\rho$  is  $n$ .

**Definition 2.1.6.** A *group algebra representation* is an extension of a group representation  $\rho$  to the group algebra  $\mathbb{C}G$  and is given by  $\tilde{\rho} : \mathbb{C}G \rightarrow M_n(\mathbb{C})$ , where

$$\tilde{\rho} \left( \sum \beta_i g_i \right) = \sum \beta_i \rho(g_i).$$

Any degree  $n$  group representation  $\rho$  determines an  $n$  dimensional  $\mathbb{C}G$ -module (and hence a  $G$ -module)  $V \subseteq \mathbb{C}G$  with basis  $\{v_1, \dots, v_n\}$ . Any element  $v \in V$  written in terms of the basis  $v = \alpha_1 v_1 + \dots + \alpha_n v_n$  can be written as a column vector  $[\alpha_i]$ . Then the module action is matrix multiplication given by  $g \cdot v := \rho(g)[\alpha_i] \in V$ . Similarly, group algebra representations determine  $\mathbb{C}G$ -modules. Since the representation determines the module, we also refer to the module  $V$  as a representation.

**Definition 2.1.7.** A representation is *irreducible* if its module is simple.

**Definition 2.1.8.** The *character*  $\chi$  of a representation  $\rho$  of  $G$  is the matrix trace of the image of the representation in  $GL_n(\mathbb{C})$ . That is,  $\chi(g) = \text{trace}(\rho(g))$ . The *degree* of  $\chi$  is the degree of  $\rho$ , and if  $\rho$  is one-dimensional, then we call  $\chi$  a linear character. If  $\rho$  is irreducible, then we also call  $\chi$  an irreducible character.

**Definition 2.1.9.** Let  $V$  be a representation of the group  $G$ , with character  $\chi$ , and  $m \geq 0$ . Then the  $m^{\text{th}}$  *Frobenius-Schur indicator* of  $V$  is given by

$$\nu_m(V) = \nu_m(\chi) := \chi \left( \frac{1}{|G|} \sum_{g \in G} g^m \right) = \frac{1}{|G|} \sum_{g \in G} \chi(g^m) \in \mathbb{C}.$$

This definition also clearly extends to representations of a group algebra, but to define the Frobenius-Schur indicators on a Hopf algebra such as  $D(G)$ , we must first extend the notion of a representation.

## 2.2. Representations and Indicators of $D(G)$

For a finite group  $G$ , [KSZ2] gave a formula for the higher Frobenius-Schur indicators of  $D(G)$ , which we include at the end of this section. The indicators are defined on representations of  $D(G)$ , and since any representation can be built from irreducible representations, we must be able to find all the irreducible representations of  $D(G)$  before we can give the formula for finding the higher indicators.

To find the irreducible representations of  $D(G)$ , we do the following:

1. Choose one  $g$  in each distinct conjugacy class of  $G$ .
2. For this  $g$ , let  $C_G(g)$  be the centralizer of  $g \in G$ .
3. Let  $V$  be a simple  $\mathbb{C}C_G(g)$ -module corresponding to an irreducible representation of  $C_G(g)$ .
4. Define  $\hat{V} = \mathbb{C}G \otimes_{\mathbb{C}C_G(g)} V$ , the induced module. Note:  $\hat{V}$  is a  $\mathbb{C}G$ -module, but not simple as a  $\mathbb{C}G$ -module.
5. Define a specific action of  $\mathbb{C}^G = (\mathbb{C}G)^*$  on  $\hat{V}$  so then  $\hat{V}$  is a  $D(G)$ -module. (See below)
6. With this action  $\hat{V}$  is a simple  $D(G)$ -module (or corresponds to an irreducible

representation) and all irreducible representations for  $D(G)$  correspond to one of these  $\hat{V}$ 's.

Steps 4, 5, and 6 are explicitly described in the following lemma.

**Lemma 2.2.1.** [KMM] *Let  $H = D(G)$  be the Drinfel'd double of  $G$ , and let  $C_G(g)$  be the centralizer of  $g \in G$ . Let  $V$  be a left  $\mathbb{C}C_G(g)$ -module, and let  $\hat{V} = \mathbb{C}G \otimes_{\mathbb{C}C_G(g)} V$ . Then  $\hat{V}$  is a left  $H$ -module, via the action*

$$(p_h \bowtie x) \cdot [y \otimes v] = \delta_{xygy^{-1}, h} xy \otimes v$$

for  $h \in G$ ,  $x, y \in C_G(g)$ , and  $v \in V$ . Furthermore, if  $V$  is a simple left  $\mathbb{C}C_G(g)$ -module, then  $\hat{V}$  is a simple left  $H$ -module. Conversely every simple left  $H$ -module is isomorphic to  $\hat{V}$  for some simple module  $V$  of  $\mathbb{C}C_G(g)$ , where  $g$  ranges over a choice of one element in each conjugate class of  $G$ .

For a finite dimensional semisimple Hopf algebra  $H$ , since  $\Delta$  is coassociative, we may define

$$\Delta^2(h) = (\Delta \otimes \text{id}) \circ \Delta(h) = (\text{id} \otimes \Delta) \circ \Delta(h) = \sum h_1 \otimes h_2 \otimes h_3$$

and inductively define  $\Delta^n(h) = (\Delta \otimes \text{id}) \circ \Delta^{n-1}(h) = \sum h_1 \otimes \cdots \otimes h_{n+1}$  for  $n \geq 2$ . We also define  $h^{[n]} = m^{n-1} \circ \Delta^{n-1}(h) = \sum h_1 \cdots h_n$  for any  $h \in H$ .

Recall an integral in a Hopf algebra is a non-zero invariant under multiplication, that is  $t \neq 0 \in H$  is an *integral* if  $ht = \epsilon(h)t = th$  for all  $h \in H$ . Since  $H$  is semisimple, by Maschke's theorem  $\epsilon(t) \neq 0$ . We will let  $\Lambda$  denote the unique integral of  $H$  with  $\epsilon(\Lambda) = 1$ .

It is not difficult to see that when  $H = D(G)$ ,

$$(2.2.2) \quad \Lambda := p_1 \bowtie \left( \frac{1}{|G|} \sum_{g \in G} g \right) = \frac{1}{|G|} \sum_{g \in G} p_1 \bowtie g,$$

For an example, see [Mo, Thm. 10.3.12].

Using the definition of  $h^{[m]}$  above, we see

$$\Lambda^{[m]} := \sum \Lambda_1 \Lambda_2 \cdots \Lambda_m.$$

Now we are ready to give the definition of higher indicators on representations of the Drinfel'd double of a group.

**Definition 2.2.3.** Given a simple  $D(G)$ -module  $V$  and its character  $\chi$  the  $m^{\text{th}}$  *Frobenius-Schur indicator* of  $V$  (or  $\chi$ ) is

$$\nu_m(V) = \nu_m(\chi) = \chi(\Lambda^{[m]}), \quad m \in \mathbb{N}.$$

In order to work with these indicators we need a little more notation and some formulas for  $\Lambda^{[m]}$ .

Our notation for the following definition is a modification of the notation used in [KSZ2, 7.2].

**Definition 2.2.4.** Let  $G$  be a finite group. For any  $x, y \in G$  and  $m \in \mathbb{N}$ , define

$$\begin{aligned} G_m(g) &= \left\{ x \in G : \prod_{j=0}^{m-1} x^{-j} g x^j = 1 \right\} \\ G_m(g, y) &= \left\{ x \in G : \prod_{j=0}^{m-1} x^{-j} g x^j = 1 \text{ and } x^m = y \right\} \\ z_m(g, y) &= |G_m(g, y)|. \end{aligned}$$

From [KSZ2] we are taking  $F = G$  and  $k = 1$ , which makes our  $G_m(g, y)$  and  $z_m(g, y)$  precisely their  $G_{m,1}(g, y)$  and  $z_{m,1}(g, y)$ .

**Proposition 2.2.5.** [KSZ2, 7.3] *Let  $G$  be a finite group and let  $\Lambda$  be the integral of  $D(G)$  in Equation 2.2.2. Then*

$$\begin{aligned} \Lambda^{[m]} &= \frac{1}{|G|} \sum_{g, y \in G} z_m(g, y) p_g \bowtie y \\ &= \frac{1}{|G|} \sum_{\substack{x \in G_m(g), \\ g \in G}} p_g \bowtie x^m. \end{aligned}$$

**Corollary 2.2.6.** [KSZ2, 7.4] *Let  $\chi$  be an irreducible character of  $D(G)$  induced from an irreducible character  $\eta$  of  $C_G(u)$  as described in Definition 2.1.8 and Lemma 2.2.1. Then*

$$\begin{aligned} \nu_m(\chi) &= \frac{1}{|G|} \sum_{\substack{x \in G_m(g), \\ g \in G}} \chi(p_g \bowtie x^m) \\ &= \frac{1}{|C_G(u)|} \sum_{y \in C_G(u)} z_m(u, y) \eta(y). \end{aligned}$$

**Remark 2.2.7.** It is worth noting that when  $u = \text{id}$ , the identity element, the centralizer  $C_G(u)$  is all of  $G$ . Thus if  $\chi$  is an irreducible character of  $D(G)$  induced from an irreducible character  $\eta$  of  $C_G(u) = G$ , then the formula in Corollary 2.2.6 becomes

$$\nu_m(\chi) = \frac{1}{|G|} \sum_{y \in G} z_m(\text{id}, y) \eta(y) = \frac{1}{|G|} \sum_{h \in G} \eta(h^m) = \nu_m(\eta).$$

where the last equivalence comes from Definition 2.1.9 and  $\nu_m(\eta)$  is an indicator of  $G$ . Thus any character of  $D(G)$  trivially induced from a character  $\eta$  of  $C_G(\text{id}) = G$  has the same indicator values as  $\eta$ .

### 2.3. Extending Group Automorphisms to $D(G)$

We now describe how a group automorphism of  $G$  can be extended to a Hopf automorphism on  $D(G)$ .

**Lemma 2.3.1.** For a group  $G$ , an automorphism  $\sigma : G \rightarrow G$  extends to an automorphism of the group algebra  $kG$  via

$$\sigma\left(\sum \alpha_g g\right) = \sum \alpha_g \sigma(g).$$

*Proof.* Multiplication is preserved since  $\sigma(\alpha g \cdot \beta h) = \sigma(\alpha \beta gh) = \alpha \beta \sigma(gh) = \alpha \beta \sigma(g) \sigma(h) = \alpha \sigma(g) \beta \sigma(h) = \sigma(\alpha g) \cdot \sigma(\beta h)$ .  $\square$

**Lemma 2.3.2.** The  $\sigma$  of Lemma 2.3.1 further extends to the dual of  $kG$ , via for  $f \in G^k$ , (where  $f : kG \rightarrow k$ ),  $\sigma(f(g)) = f(\sigma(g))$ . When we consider the basis elements of  $G^k$ , namely  $p_x$  where  $x \in G$ , we see that

$$\sigma(p_x) = p_{\sigma^{-1}(x)}.$$

*Proof.*  $\sigma(p_x(g)) = p_x(\sigma(g)) = \begin{cases} 1 & x = \sigma(g) \\ 0 & x \neq \sigma(g) \end{cases} = \begin{cases} 1 & \sigma^{-1}(x) = g \\ 0 & \sigma^{-1}(x) \neq g \end{cases} = p_{\sigma^{-1}(x)}(g),$   
 $\forall g \in G$ .

Thus since  $p_x \cdot p_y = \delta_{x,y} p_x$ , we confirm that this extension does indeed preserve multiplication on  $G^k$ :

$\sigma(p_x \cdot p_y) = \sigma(\delta_{x,y} p_x) = \delta_{x,y} p_{\sigma^{-1}(x)}$  and  
 $\sigma(p_x) \cdot \sigma(p_y) = p_{\sigma^{-1}(x)} \cdot p_{\sigma^{-1}(y)} = \delta_{\sigma^{-1}(x), \sigma^{-1}(y)} p_{\sigma^{-1}(x)}$  and since  
 $x = y \Leftrightarrow \sigma^{-1}(x) = \sigma^{-1}(y)$ , that means  $\delta_{x,y} = \delta_{\sigma^{-1}(x), \sigma^{-1}(y)}$ , thus multiplication is preserved.  $\square$

We now further extend  $\sigma$  to  $D(G)$  in the following way:

**Definition 2.3.3.** Let  $\gamma_\sigma : D(G) \rightarrow D(G)$  be defined as:

$$\gamma_\sigma(p_x \bowtie g) = \sigma(p_x) \bowtie \sigma^{-1}(g) = p_{\sigma^{-1}(x)} \bowtie \sigma^{-1}(g).$$

**Definition 2.3.4.** An automorphism of a Hopf algebra  $H$  is a map  $\gamma : H \rightarrow H$  that preserves the multiplication, unit, comultiplication, counit, and antipode of  $H$ . That is for all  $g, h$  in  $H$ ,  $\gamma(g \cdot h) = \gamma(g) \cdot \gamma(h)$ ,  $(\gamma \otimes \gamma)(\Delta(h)) = \Delta(\gamma(h))$ , and  $\gamma(S(h)) = S(\gamma(h))$ .

**Proposition 2.3.5.**  $\gamma_\sigma$  is a Hopf automorphism of  $D(G)$ .

*Proof.* 1) Using the definition of multiplication in  $D(G)$  given in Definition 2.1.4, we see that

$$\begin{aligned} \gamma_\sigma((p_x \bowtie g) \cdot (p_y \bowtie h)) &= \gamma_\sigma(\delta_{x,gyg^{-1}} p_x \bowtie gh) \\ &= \delta_{x,gyg^{-1}} p_{\sigma^{-1}(x)} \bowtie \sigma^{-1}(gh). \end{aligned}$$

Since  $x = gyg^{-1}$  if and only if  $\sigma^{-1}(x) = \sigma^{-1}(gyg^{-1}) = \sigma^{-1}(g)\sigma^{-1}(y)\sigma^{-1}(g^{-1}) = \sigma^{-1}(g)\sigma^{-1}(y)(\sigma^{-1}(g))^{-1}$ , this means  $\delta_{x,gyg^{-1}} = \delta_{\sigma^{-1}(x), \sigma^{-1}(g)\sigma^{-1}(y)(\sigma^{-1}(g))^{-1}}$ . Thus

$$\begin{aligned} \gamma_\sigma((p_x \bowtie g) \cdot (p_y \bowtie h)) &= \gamma_\sigma(\delta_{x,gyg^{-1}} p_x \bowtie gh) \\ &= \delta_{x,gyg^{-1}} p_{\sigma^{-1}(x)} \bowtie \sigma^{-1}(gh) \\ &= \delta_{\sigma^{-1}(x), \sigma^{-1}(g)\sigma^{-1}(y)(\sigma^{-1}(g))^{-1}} p_{\sigma^{-1}(x)} \bowtie \sigma^{-1}(g)\sigma^{-1}(h) \\ &= (p_{\sigma^{-1}(x)} \bowtie \sigma^{-1}(g)) \cdot (p_{\sigma^{-1}(y)} \bowtie \sigma^{-1}(h)) \\ &= \gamma_\sigma(p_x \bowtie g) \cdot \gamma_\sigma(p_y \bowtie h). \end{aligned}$$

Thus  $\gamma_\sigma$  preserves multiplication. Clearly,  $\gamma_\sigma$  preserves the unit.

2) Using the definition of comultiplication in  $D(G)$  given in Definition 2.1.4, we see that

$$\begin{aligned}
\gamma_\sigma(\Delta(p_g \bowtie x)) &= (\gamma_\sigma \otimes \gamma_\sigma)\left(\sum_{h \in G} (p_h \bowtie x) \otimes (p_{h^{-1}g} \bowtie x)\right) \\
&= \sum_{h \in G} \gamma_\sigma(p_h \bowtie x) \otimes \gamma_\sigma(p_{h^{-1}g} \bowtie x) \\
&= \sum_{h \in G} (p_{\sigma^{-1}(h)} \bowtie \sigma^{-1}(x)) \otimes (p_{\sigma^{-1}(h^{-1}g)} \bowtie \sigma^{-1}(x)) \\
&= \sum_{h \in G} (p_{\sigma^{-1}(h)} \bowtie \sigma^{-1}(x)) \otimes (p_{(\sigma^{-1}(h))^{-1}\sigma^{-1}(g)} \bowtie \sigma^{-1}(x)) \\
&= \sum_{k \in G} (p_k \bowtie \sigma^{-1}(x)) \otimes (p_{k^{-1}\sigma^{-1}(g)} \bowtie \sigma^{-1}(x)) \\
&= \Delta(p_{\sigma^{-1}(g)} \bowtie \sigma^{-1}(x)) \\
&= \Delta(\gamma_\sigma(p_g \bowtie x)),
\end{aligned}$$

where in the fifth equality  $k = \sigma^{-1}(h)$ .

Thus  $\gamma_\sigma$  preserves comultiplication. Clearly,  $\gamma_\sigma$  preserves the counit.

3) Finally, using the definition of the antipode of  $D(G)$  given in Definition 2.1.4, we see that

$$\begin{aligned}
\gamma_\sigma(S(p_x \bowtie g)) &= \gamma_\sigma(p_{g^{-1}x^{-1}g} \bowtie g^{-1}) \\
&= p_{\sigma^{-1}(g^{-1}x^{-1}g)} \bowtie \sigma^{-1}(g^{-1}) \\
&= p_{\sigma^{-1}(g^{-1})\sigma^{-1}(x^{-1})\sigma^{-1}(g)} \bowtie \sigma^{-1}(g^{-1}) \\
&= p_{(\sigma^{-1}(g))^{-1}(\sigma^{-1}(x))^{-1}\sigma^{-1}(g)} \bowtie (\sigma^{-1}(g))^{-1} \\
&= S(p_{\sigma^{-1}(x)} \bowtie \sigma^{-1}(g)) \\
&= S(\gamma_\sigma(p_x \bowtie g)).
\end{aligned}$$

Thus  $\gamma_\sigma$  preserves the antipode, and  $\gamma_\sigma$  is a Hopf automorphism.  $\square$

**Lemma 2.3.6.** *Let  $\chi$  be an irreducible character of  $G$  and  $\sigma \in \text{Aut}(G)$ . Define the map  $\chi^\sigma$  to be  $\chi^\sigma(g) = \chi(\sigma(g))$  for  $g \in G$ . Then  $\chi^\sigma$  is also an irreducible character of  $G$ .*

This fact is well known in group theory.  $\chi^\sigma$  is called quasi-equivalent to  $\chi$ .

Lemma 2.3.6 also extends to automorphism on Hopf algebras.

**Corollary 2.3.7.** *Let  $\chi$  be an irreducible character of a Hopf algebra  $H$  and  $\gamma \in \text{Aut}(H)$ . Define the map  $\chi^\gamma$  to be  $\chi^\gamma(h) = \chi(\gamma(h))$  for  $h \in H$ . Then  $\chi^\gamma$  is also an irreducible character of  $H$ .*

**Lemma 2.3.8.** *Let  $\chi$  be an irreducible character of  $D(G)$  induced from an irreducible character  $\eta$  of the centralizer  $C_G(u)$ , where  $u \in G$ . Let  $\sigma \in \text{Aut}(G)$  and  $\gamma_\sigma \in \text{Aut}(D(G))$ . Then  $\chi^{\gamma_\sigma}$  is an irreducible character of  $D(G)$  which is quasi-equivalent to one induced from an irreducible character of the centralizer  $C_G(\sigma(u))$ .*

*Proof.* We see that  $\sigma(C_G(g)) = C_G(\sigma(g))$  since if  $g, x \in G$ , then  $x \in \sigma(C_G(g)) \Leftrightarrow \sigma^{-1}(x) \in C_G(g) \Leftrightarrow \sigma^{-1}(x)g = g\sigma^{-1}(x) \Leftrightarrow x\sigma(g) = \sigma(g)x \Leftrightarrow x \in C_G(\sigma(g))$ . Thus we have equality.



Since  $\sigma$  is a group automorphism,  $C_G(g) \cong \sigma(C_G(g)) = C_G(\sigma(g))$  and thus any irreducible character of  $C_G(g)$  is quasi-equivalent to an irreducible character of  $C_G(\sigma(g))$ . That is, if  $\eta$  is an irreducible character of  $C_G(g)$ , then  $\eta^\sigma$  is an irreducible character of  $C_G(\sigma(g))$ . Thus, if  $\chi$  is an irreducible character of  $D(G)$  induced from  $\eta$ , then  $\chi^{\gamma^\sigma}$  is an irreducible character of  $D(G)$  induced from  $\eta^\sigma$ .  $\square$

**Example 2.3.9.** Consider when  $G = S_n$ . Let  $\sigma \in \text{Aut}(S_n)$  and  $\chi$  be an irreducible character of  $D(S_n)$ . If  $n \neq 6$ , then  $\sigma$  must be an inner automorphism, meaning it is defined by conjugation. Since conjugation preserves the conjugacy classes, we see that  $\chi^\sigma = \chi$  for  $n \neq 6$ .

This however is not true for  $G = S_6$ . Below we explicitly provide the outer automorphism of  $S_6$  from [R Corollary 7.13].

**Definition 2.3.10.** Define  $\sigma \in \text{Aut}(S_6)$  by

$$\begin{aligned} (1\ 2) &\mapsto (1\ 5)(2\ 3)(4\ 6), \\ (1\ 3) &\mapsto (1\ 4)(2\ 6)(3\ 5), \\ (1\ 4) &\mapsto (1\ 3)(2\ 4)(5\ 6), \\ (1\ 5) &\mapsto (1\ 2)(3\ 6)(4\ 5), \\ (1\ 6) &\mapsto (1\ 6)(2\ 5)(3\ 4). \end{aligned}$$

A routine but long calculation shows that  $\sigma^2 = 1$ . From this definition we can compute that

$$\begin{aligned} (1\ 2)(3\ 4)(5\ 6) &\mapsto (2\ 3), \\ (1\ 2)(3\ 4) &\mapsto (1\ 4)(5\ 6), \\ (1\ 2\ 3) &\mapsto (1\ 3\ 6)(2\ 5\ 4), \\ (1\ 2\ 3)(4\ 5\ 6) &\mapsto (1\ 6\ 3), \\ (1\ 2\ 3)(4\ 5) &\mapsto (1\ 4\ 6\ 5\ 3\ 2), \\ (1\ 2\ 3\ 4\ 5\ 6) &\mapsto (1\ 5\ 6)(2\ 4), \\ (1\ 2\ 3\ 4) &\mapsto (2\ 6\ 3\ 5), \\ (1\ 2\ 3\ 4)(5\ 6) &\mapsto (1\ 4)(2\ 5\ 3\ 6), \\ (1\ 2\ 3\ 4\ 5) &\mapsto (1\ 2\ 3\ 4\ 5). \end{aligned}$$

So we see that the product of three transpositions maps to a single transposition and vice-a-versa, 3-cycles map to a product of two 3-cycles and vice-a-versa, 6-cycles map to a product of a transposition and a 3-cycle and vice-a-versa, and all other cycle types map to their own type. We will use this fact later to prove Proposition 3.3.3.

We end this section with a final Proposition, which proves the higher indicators of two quasi-equivalent characters are equal, and a question.

**Proposition 2.3.11.** *Let  $\chi$  be an irreducible character of  $D(G)$  and let  $\sigma \in \text{Aut}(D(G))$ . Then  $\nu_m(\chi^\sigma) = \nu_m(\chi)$  for all  $m \in \mathbb{N}$ .*

*Proof.* Recall that  $\Lambda$  given in Equation 2.2.2 is the unique integral of  $D(G)$ , and so  $\Lambda^\sigma = \sigma(\Lambda) = \Lambda$ . Since  $\sigma$  also preserves the coalgebra structure of  $D(S_n)$ ,  $(\Lambda^{[n]})^\sigma = (\Lambda^\sigma)^{[n]} = \Lambda^{[n]}$ . Thus when we look at the original definition of the higher indicators

as given in Definition 2.2.3 we see that  $\nu_m(\chi^\sigma) = \chi^\sigma(\Lambda^{[m]}) = \chi(\sigma(\Lambda^{[m]})) = \chi(\Lambda^{[m]}) = \nu_m(\chi)$ .  $\square$

**Question 2.3.12.** *Does every Hopf automorphism of  $D(S_n)$  for  $n \geq 5$  come from an automorphism  $\sigma \in \text{Aut}(G)$  as in Lemma 2.3.8?*

### 3. INDICATOR EQUIVALENCES

In this section we explore when higher Frobenius-Schur indicators are equivalent. We first discuss what formulas we used to compute all the indicators of  $D(S_n)$  in Section 3.1. Then in Section 3.2, since all of the indicators we calculated were integers, we discuss results showing we need only consider a smaller subset of all possible higher indicators  $\nu_m$  of a character to know them all. In Section 3.3 we discuss when all higher indicators of different characters are correspondingly equivalent, and then focus on when an indicator is zero valued in Section 3.4. We finish this section by trying to answer a question that naturally arises from a proposition in Section 3.4.

#### 3.1. A More Computable Formula for $\nu_m$

We first computed indicators in GAP using the formula in Corollary 2.2.6. However, these computations were very time consuming, and we were only able to compute the indicators for  $D(S_3)$  through  $D(S_6)$  using this method. Attempting to compute the indicators for  $D(S_7)$  overloaded our computer. We needed a more efficient formula or way of programming in order to compute the indicators for the double of larger symmetric groups.

Recently a different formula was given in [IMM] for calculating the higher indicators of other Hopf algebras. We will use a variation of this formula which gives a more efficient computation.

Recall from Definition 2.2.4 that

$$G_m(g, y) = \left\{ x \in G : \prod_{j=0}^{m-1} x^{-j} g x^j = 1 \text{ and } x^m = y \right\}.$$

We fix  $g = u$  in a conjugacy class. Then as noted in [IMM],

$$G_m(u, y) = \{h \in G \mid h^m = y, (uh)^m = h^m\}.$$

That is  $(uh)^m = h^m$  if and only if  $\prod_{j=0}^{m-1} h^{-j} u h^j = 1$ .

Now let  $\eta$  be a character for an irreducible representation of  $C_G(u)$ , where as before  $C_G(u)$  is the centralizer of our element  $u$ . When calculating  $\nu_m(\chi)$  as in Corollary 2.2.6, [IMM] further noted that the first condition of  $G_m(u, y)$  is superfluous. That is,  $(uh)^m = h^m$  implies  $h^m \in C_G(u)$ . Thus instead of summing over the centralizer, we can sum over a new set.

**Definition 3.1.1.** For a group  $G$ , define  $\tilde{G}_m(u) := \{h \in G \mid (uh)^m = h^m\}$ .

**Corollary 3.1.2.** [IMM] *For a group  $G$ , a fixed element  $u$  in a conjugacy class, and an irreducible character  $\eta$  of  $C_G(u)$ , the formula for the  $m^{\text{th}}$  Frobenius-Schur indicator of a character  $\chi$  of  $D(G)$  induced from  $\eta$  as given in Corollary 2.2.6 is equivalent to*

$$\nu_m(\chi) = \frac{1}{|C_G(u)|} \sum_{h \in \tilde{G}_m(u)} \eta(h^m).$$

In order to make the formula in Corollary 3.1.2 more efficient when using GAP, we consider the elements  $h^m$ . There may be many different  $h \in \tilde{G}_m(u)$  that have the same  $h^m$ , so rather than summing over  $h$ , we could sum over  $h^m$ . But even distinct  $h^m$  may be in the same conjugacy class of  $C_G(u)$ , which means the value of  $\eta(h^m)$  will be the same, so we could sum over the conjugacy classes of  $C_G(u)$ . Thus to avoid summing the same value more than once we give the following definitions and notation.

Let  $\text{Conj}_{C_G(u)}$  be the set of conjugacy classes of  $C_G(u)$ , and  $\mathfrak{R}_{C_G(u)}$  be a fixed set of conjugacy class representatives - that is, a set of elements in  $C_G(u)$  such that each element is from a different conjugacy class.

For example, consider  $C_{S_4}((12)(34)) = \langle (12), (13)(24), (34) \rangle \cong D_8$ . Then  $\text{Conj}_{C_{S_4}((12)(34))} = \{ [()], [(12), (34)], [(12)(34)], [(13)(24), (14)(23)], [(1324), (1423)] \}$  and we may choose  $\mathfrak{R}_{C_{S_4}((12)(34))} = \{ (), (12), (12)(34), (13)(24), (1324) \}$ .

**Definition 3.1.3.** Let  $\mathfrak{R}_{C_G(u)}$  be a fixed set of conjugacy class representatives of  $C_G(u)$ . Define  $\tilde{G}_m^m(u)$  to be the set of all  $y \in \mathfrak{R}_{C_G(u)}$  for which there exists an  $h \in \tilde{G}_m(u)$  such that  $h^m$  is in the conjugacy class of  $y$  in  $C_G(u)$ .

To construct the set  $\tilde{G}_m^m(u)$  from the set  $\tilde{G}_m(u)$  we would:

- (1) raise all elements  $h \in \tilde{G}_m(u)$  to the  $m^{\text{th}}$  power;
- (2) organize these new elements into their conjugacy classes; and
- (3) choose one representative from each of these conjugacy classes.

The set  $\tilde{G}_m^m(u)$  is the collection of all these conjugacy class representatives. Another description of  $\tilde{G}_m^m(u)$  will be given in Lemma 3.1.5 below.

**Definition 3.1.4.** Let  $y \in C_G(u)$ . Define  $\Gamma_m(u, y)$  to be the number of  $h \in G$  such that  $h \in \tilde{G}_m(u)$  and  $h^m$  is in the conjugacy class of  $y$  in  $C_G(u)$ , that is:

$$\Gamma_m(u, y) := |\{h \in G \mid (uh)^m = h^m \text{ and } h^m \in \text{cl}_{C_G(u)}(y)\}|,$$

where  $\text{cl}_{C_G(u)}(y)$  denotes the conjugacy class of  $y$  in  $C_G(u)$ .

Note that for  $y_1$  and  $y_2$  in the same conjugacy class of  $C_G(u)$ ,  $\Gamma_m(u, y_1) = \Gamma_m(u, y_2)$ .

**Lemma 3.1.5.** Using  $\Gamma_m(u, y)$  we see:

$$\tilde{G}_m^m(u) = \{y \in \mathfrak{R}_{C_G(u)} \mid \Gamma_m(u, y) \neq 0\}.$$

**Proposition 3.1.6.** Let  $u$  be a fixed representative of a conjugacy class of  $G$ , and let  $\eta$  be an irreducible character of  $C_G(u)$ . Then the  $m^{\text{th}}$  Frobenius-Schur indicator of a character  $\chi$  of  $D(G)$  induced from  $\eta$  is given by

$$\nu_m(\chi) = \frac{1}{|C_G(u)|} \sum_{y \in \tilde{G}_m^m(u)} \Gamma_m(u, y) \eta(y).$$

*Proof.* Using the formula in Corollary 3.1.2 and Definitions 3.1.3 and 3.1.4, we have

$$\begin{aligned}
 \nu_m(\chi) &= \frac{1}{|C_G(u)|} \sum_{h \in \tilde{G}_m(u)} \eta(h^m) \\
 &= \frac{1}{|C_G(u)|} \sum_{y \in \mathfrak{R}_{C_G(u)}} \Gamma_m(u, y) \eta(y) \\
 &= \frac{1}{|C_G(u)|} \sum_{y \in \tilde{G}_m^m(u)} \Gamma_m(u, y) \eta(y)
 \end{aligned}$$

□

The formula in Proposition 3.1.6 is the function we used to calculate all the higher indicators for  $D(S_n)$  for  $n \leq 10$ . Further discussion on how we translated this formula into GAP functions is found in Section 5.

### 3.2. Equivalent Indicator Values

There are an infinite number of higher indicators  $\nu_m$  for a single character  $\chi$  of  $D(G)$ , but in order to find all of them, it turns out that we only need to calculate a small set of them. We only need to find  $\nu_d$  where  $d$  is a divisor of the exponent of the group  $G$ . Recall, the *exponent* of a group  $G$  is the smallest positive integer  $e$  such that  $g^e = 1$  for all  $g \in G$ . We write  $e = \exp(G)$  for the exponent of  $G$ . Thus all of our indicator calculations found in Section 4 only included such  $\nu_d$ .

[IMM] showed that if  $\nu_d(\chi)$  is an integer for every character  $\chi$ , then in fact each  $\nu_m(\chi)$  is equal to some  $\nu_d(\chi)$ , and so all the indicators will be found by just finding the  $\nu_d$ .

**Theorem 3.2.1.** [IMM] *Let  $\chi$  be an irreducible character of  $D(G)$ , induced from  $\eta$  an irreducible character on  $C_G(u)$ , for  $u \in G$  fixed. Let  $e = \exp(G)$ , the exponent of the group  $G$ , and say  $m \in \mathbb{N}$  and  $d = \gcd(m, e)$ . Then*

1. *If  $\nu_d(\chi) \in \mathbb{Z}$ , for all  $\chi$  induced from characters on  $C_G(u)$ , then  $\nu_m(\chi) \in \mathbb{Z}$ , for all  $\chi$ .*
2. *If  $\nu_d(\chi) \in \mathbb{Z}$ ,  $m = dk$  and  $(k, e) = 1$ , then  $\nu_{dk}(\chi) = \nu_d(\chi)$ .*
- 2'. *If  $\nu_d(\chi) \in \mathbb{Z}$ , and  $m = dk$ , then  $\nu_{dk}(\chi) = \nu_d(\chi)$ .*

It should be noted that the equivalence of 2 and 2' in Theorem 3.2.1 is due to Richard Ng.

### 3.3. I-equivalent Irreducible Characters

When first calculating all the indicators, we noticed that many distinct irreducible characters have the same set of indicator values. We decided to create equivalence classes of characters to avoid repeating the same indicators over and over again in our tables.

**Definition 3.3.1.** Two characters  $\chi$  and  $\eta$  of  $D(G)$  are indicator equivalent or *I-equivalent* if  $\nu_m(\chi) = \nu_m(\eta)$  for all positive integers  $m$ . Conversely, two characters  $\chi$  and  $\eta$  of  $D(G)$  are *I-inequivalent* if there exists an  $m$  for which  $\nu_m(\chi) \neq \nu_m(\eta)$ .

As we saw at the end of Section 2 in Corollary 2.3.11, given an automorphism  $\sigma$  of  $D(G)$ , two quasi-equivalent characters  $\chi$  and  $\chi^\sigma$  are I-equivalent.

I-equivalent is an equivalence relationship, so all the irreducible characters of  $D(S_n)$  can be collected by their *I-equivalence class*. Most irreducible character I-equivalence

classes of  $D(S_n)$  only contain characters induced from the same centralizer  $C_{S_n}(u)$ , however in a few instances this is not the case.

**Definition 3.3.2.** We say that an irreducible character I-equivalence class is

- 1) *homogenous* if all the characters in that class are induced from the same centralizer  $C_G(u)$ , or
- 2) *mixed* if it contains characters induced from different centralizers of  $G$ .

**Proposition 3.3.3.**

(1) *The irreducible character I-equivalency classes of  $D(S_5)$ ,  $D(S_7)$ ,  $D(S_8)$ ,  $D(S_9)$ , and  $D(S_{10})$  are all homogenous.*

(1') *If the irreducible characters  $\chi_1$  and  $\chi_2$  of  $D(S_n)$  have the same indicator values for all  $m$ , then  $\chi_1$  and  $\chi_2$  are induced from the same centralizer  $C_{S_n}(u)$  for  $n = 5$ ,  $7 \leq n \leq 10$ .*

(2) *For  $D(S_6)$ , the only irreducible character I-equivalence classes that are mixed may be as expected from the outer automorphism on  $S_6$ .*

*Proof.* The first statement is proven by observation of the I-equivalency classes as recorded in Section 4 and Appendix B of this paper. The second statement considers the outer automorphism  $\sigma$  of  $S_6$  that has order 2 given in [R] and explicitly provided in Definition 2.3.10. This outer automorphism maps transpositions to the product of three transpositions, 3-cycles to the product of two 3-cycles, 6-cycles to the product of a 3-cycle and a transposition, and all other elements to their same cycle type. When we look at the irreducible character I-equivalency classes we find the characters induced from  $C_{S_6}((1, 2))$  are mixed with or I-equivalent to characters induced from  $C_{S_6}((1, 2)(3, 4)(5, 6))$ . Characters induced from  $C_{S_6}((1, 2, 3))$  are mixed with or I-equivalent to characters induced from  $C_{S_6}((1, 2, 3)(4, 5, 6))$ . Characters induced from  $C_{S_6}((1, 2, 3)(4, 5))$  are mixed with or I-equivalent to characters induced from  $C_{S_6}((1, 2, 3, 4, 5, 6))$ . All other characters, induced from other centralizers are in homogeneous irreducible character I-equivalence classes. This is consistent with the outer automorphism of  $S_6$  and with the fact that an irreducible character induced from  $C_{S_6}(u)$  is I-equivalent to another one induced from  $C_{S_6}(\sigma(u))$  as was shown in Lemma 2.3.8 and Proposition 2.3.11.  $\square$

We also note that the mixed I-equivalence classes of  $D(S_3)$  and  $D(S_4)$  are not unexpected due to the small sizes of  $S_3$  and  $S_4$ . Proposition 3.3.3 raises a natural question.

**Question 3.3.4.** *Is it true for all  $n \geq 5$ ,  $n \neq 6$ , that all irreducible character I-equivalency classes of  $D(S_n)$  are homogenous?*

We note that this question is related to Question 2.3.12.

We tried to see if there was a connection between characters being I-equivalent and the centralizers from which they were induced. This is why in Section 4 we give details about the centralizer groups, specifically noting if they are abelian or not. Recall that all of the irreducible characters of an abelian group  $G$  are linear.

**Lemma 3.3.5.** *If  $n = 3$  or  $4$ , then the irreducible characters of  $D(S_n)$  induced from the same centralizer  $C_{S_n}(u)$  are I-equivalent if and only if  $C_{S_n}(u)$  is abelian.*

*Proof.* By observation, as recorded in Section 4, Sections 1 and 2.  $\square$

Lemma 3.3.5, however, does not extend to the irreducible characters of larger doubles, such as  $D(S_5)$ , as we see in the following example.

**Example 3.3.6.**

(1) In  $S_5$ , let  $u_4 = (1, 2, 3)$ . Then its centralizer is  $C_{S_5}(u_4) \cong C_6$ , the cyclic group of order 6, which is abelian. However, the irreducible characters induced from  $u_4$  are broken into two homogenous I-equivalency classes. Thus an abelian centralizer does not imply all irreducible characters induced from that centralizer are in the same I-equivalency class.

(2) Now let  $u_5 = (1, 2, 3)(4, 5)$ . The centralizers of  $u_4$  and  $u_5$  are identical, since  $C_{S_5}(u_5) \cong C_6$ . Yet, all the irreducible characters induced from  $u_5$  are in the same single homogenous I-equivalency class. Thus irreducible characters induced from isomorphic centralizers are not necessarily I-equivalent, nor will those isomorphic centralizers have the same number of irreducible character I-equivalency classes.

We must also consider the set we sum over when computing the higher Frobenius-Schur indicators, that is  $\tilde{G}_m^m(u)$ . From our computations, we noticed that if the sets  $\tilde{G}_m^m(u)$  were not empty, then they contained the identity element. We also noticed that a portion of the  $\tilde{G}_m^m(u)$  contained only the identity.

**Lemma 3.3.7.** *Let  $u$  be a fixed conjugacy class representative of  $G$ . If  $C_G(u)$  is abelian, and  $\tilde{G}_m^m(u) = \emptyset$  or  $\tilde{G}_m^m(u) = \{id\}$  for each  $m$ , then all  $D(G)$  characters induced from  $C_G(u)$  will be I-equivalent.*

*Proof.* Let  $\chi_1$  and  $\chi_2$  be irreducible characters of  $D(G)$  induced from irreducible characters of  $C_G(u)$ ,  $\eta_1$  and  $\eta_2$  respectively. Since  $C_G(u)$  is abelian, that means  $\eta_1$  and  $\eta_2$  are linear characters, so  $\eta_1(id) = 1 = \eta_2(id)$ . Now for a fixed  $m$ , if  $\tilde{G}_m^m(u) = \emptyset$ , then  $\nu_m(\chi_1) = 0 = \nu_m(\chi_2)$ . If  $\tilde{G}_m^m(u) = \{id\}$ , then

$$\begin{aligned} \nu_m(\chi_1) &= \frac{1}{|C_G(u)|} \sum_{y \in \tilde{G}_m^m(u)} \Gamma_m(u, y) \eta_1(y) \\ &= \frac{1}{|C_G(u)|} \Gamma_m(u, id) \eta_1(id) \\ &= \frac{1}{|C_G(u)|} \Gamma_m(u, id) \eta_2(id) \\ &= \frac{1}{|C_G(u)|} \sum_{y \in \tilde{G}_m^m(u)} \Gamma_m(u, y) \eta_2(y) \\ &= \nu_m(\chi_2) \end{aligned}$$

Thus  $\chi_1$  and  $\chi_2$  are I-equivalent. □

We make one last observation regarding Lemma 3.3.7. We ask if Lemma 3.3.7 can be weakened by omitting the assumption that  $C_G(u)$  is abelian, since for  $D(S_n)$  where  $n \leq 10$ , all the irreducible characters induced from the same  $C_{S_n}(u)$  were I-equivalent exactly when for each  $m$ ,  $\tilde{G}_m^m(u)$  was empty or only contained the identity. We end this section with this question.

**Question 3.3.8.** *Does the condition that for each  $m$ ,  $\tilde{G}_m^m(u) = \emptyset$  or  $\tilde{G}_m^m(u) = \{id\}$  imply that  $C_G(u)$  is abelian?*

### 3.4. Zero Valued Indicators

In this section we discuss when a higher indicator of a character of  $D(S_n)$  may have a value of zero. As mentioned in Remark 2.2.7, some characters of  $D(S_n)$  are trivially induced and have the same indicator values as those of  $S_n$ . Since the higher indicators of  $S_n$  have been known for some time, we would expect some of these trivially induced characters to have zero valued indicators - which some do. So we exclude this situation (when  $u = \text{id}$ ) from our consideration of when  $\nu_m(\chi) = 0$ . Another occurrence of zero valued indicators that we ignore is when  $m = 1$ . When  $m = 1$ , all but the induced trivial character of  $S_n$  have indicator values of zero, as we would expect. Thus, when  $\chi$  is trivially induced or  $m = 1$ , we expect that  $\nu_m(\chi)$  may be zero.

When first looking over our tables of indicators, we noticed that for odd  $m$ , there were many zero valued indicators. More specifically, there were clusters of zeros coming from characters induced from the same centralizers,  $C_G(u)$ . After inspecting the sets  $\tilde{G}_m^m(u)$  that we sum over, we realized they were all empty, and that was why the indicator values were zero. Note that we have already seen significance in  $\tilde{G}_m^m(u)$  being empty in Lemma 3.3.7. Now we consider how it affects the indicator values themselves.  $\tilde{G}_m^m(u)$  is empty if and only if  $\tilde{G}_m(u)$  is empty, thus the simplest case when  $\nu_m(\chi) = 0$  is exactly when  $\tilde{G}_m(u) = \{h \in G \mid (uh)^m = h^m\} = \emptyset$ .

**Proposition 3.4.1.** *If  $u$  is an odd permutation of  $G = S_n$  and  $m$  is an odd positive integer, then  $\tilde{G}_m(u) = \emptyset$ . Thus  $\nu_m(\chi) = 0$  when  $m$  is odd and  $\chi$  is induced from an irreducible character of  $C_{S_n}(u)$  for  $u$  an odd permutation.*

*Proof.* If  $u$  is an odd permutation of  $S_n$  then  $\text{sgn}(u) = -1$ . Recall that for arbitrary permutations  $a$  and  $b$ ,  $\text{sgn}(ab) = \text{sgn}(a)\text{sgn}(b)$ . Thus we have for any  $h \in S_n$ ,  $\text{sgn}(h^m) = \text{sgn}(h)^m$ , and  $\text{sgn}((uh)^m) = \text{sgn}(uh)^m$ . If  $m$  is an odd positive integer, then  $h^m$  has the same parity as  $h$ , and  $(uh)^m$  has the same parity as  $uh$ .

Thus for any  $h \in S_n$

$$\text{sgn}((uh)^m) = \text{sgn}(uh) = \text{sgn}(u)\text{sgn}(h) = -\text{sgn}(h) \neq \text{sgn}(h) = \text{sgn}(h^m)$$

and so  $(uh)^m$  can not equal  $h^m$ . Thus  $\tilde{G}_m(u) = \emptyset$ , which means  $\nu_m(\chi) = 0$  when  $m$  is odd and  $\chi$  is induced from an irreducible character of  $C_{S_n}(u)$  for  $u$  an odd permutation.  $\square$

Certainly  $\nu_m(\chi)$  can be 0 when  $\tilde{G}_m^m(u)$  is not empty, although this appears somewhat rare when we consider all indicators of  $D(S_3)$  through  $D(S_{10})$ . First, we formally define the notion of expected zeros and unexpected zeros using Proposition 3.4.1.

**Definition 3.4.2.** We say  $\nu_m(\chi) = 0$  is an *expected zero* if  $m = 1$ , or  $\chi$  is trivially induced, or both  $u$  is an odd permutation and  $m$  is an odd positive integer. Conversely, we say  $\nu_m(\chi) = 0$  is an *unexpected zero* if  $m \neq 1$ ,  $\chi$  is not trivially induced, and  $u$  and  $m$  are not both odd. We also may refer to an unexpected zero as an *unexpected zero valued indicator*.

Next, recall that by Theorem 3.2.1, to find all possible values of indicators  $\nu_m(\chi)$ , it suffices to find  $\nu_d(\chi)$  where  $d$  divides the exponent  $e$ , provided  $\nu_d(\chi) \in \mathbb{Z}$ . Thus we say a *distinct non-trivial higher indicator* of  $D(G)$  is an indicator  $\nu_m$  such that  $m \geq 3$  and  $m$  is a divisor of the exponent of the group  $G$ . Here we consider  $\nu_1$  and

$\nu_2$  to be trivial since for any irreducible character of  $D(G)$ ,  $\nu_1$  is equal to 0 unless  $\chi$  is the trivial character and  $\nu_2$  is always equal to 1.

Finally, a *non-trivial indicator value* is the value of  $\nu_m(\chi)$  where  $\chi$  is not a trivially induced character and  $m \geq 3$ .

**Proposition 3.4.3.**

- 1)  $D(S_3)$ ,  $D(S_4)$ , and  $D(S_5)$  have no unexpected zero valued indicators.
- 2)  $D(S_6)$  has only two unexpected zero valued indicators. It has thirteen non-trivially induced I-equivalent character classes and 10 distinct non-trivial higher indicators. Thus  $D(S_6)$  has 130 non-trivial indicator values.
- 3)  $D(S_7)$  has only two unexpected zero valued indicators. It has 33 non-trivially induced I-equivalent character classes and 22 distinct non-trivial higher indicators. Thus  $D(S_7)$  has 726 non-trivial indicator values.
- 4)  $D(S_8)$  has thirteen unexpected zero valued indicators. It has 66 non-trivially induced I-equivalent character classes and 30 distinct non-trivial higher indicators. Thus  $D(S_8)$  has 1980 non-trivial indicator values.
- 5)  $D(S_9)$  has 38 unexpected zero valued indicators. It has 107 non-trivially induced I-equivalent character classes and 46 distinct non-trivial higher indicators. Thus  $D(S_9)$  has 4922 non-trivial indicator values.
- 6)  $D(S_{10})$  has 21 unexpected zero valued indicators. It has 196 non-trivially induced I-equivalent character classes and 46 distinct non-trivial higher indicators. Thus  $D(S_{10})$  has 9016 non-trivial indicator values.

*Proof.* By observation of indicator tables found in Section 4 and Appendix B.  $\square$

To summarize, 1.54% of  $D(S_6)$ 's non-trivial indicator values are unexpected zeros, 0.28% of  $D(S_7)$ 's non-trivial indicator values are unexpected zeros, 0.66% of  $D(S_8)$ 's non-trivial indicator values are unexpected zeros, 0.77% of  $D(S_9)$ 's non-trivial indicator values are unexpected zeros, and 0.23% of  $D(S_{10})$ 's non-trivial indicator values are unexpected zeros. Thus most zeros are expected.

Now in Table 3.1 below, we explicitly list the conditions in which the unexpected zeros from Proposition 3.4.3 appear. In Table 3.1:

- $S_n$  tells us which double  $D(S_n)$  has unexpected zero valued indicators,
- $u$  is the conjugacy class representative of  $S_n$  from whose centralizer the characters of  $D(S_n)$  are induced,
- $m$  refers to which higher indicator  $\nu_m$  had a zero value, and
- “no. of chars with  $\nu_m = 0$ ” is the number of I-equivalent character classes for which the indicator is zero when  $u$  and  $m$  are as specified. In this column “all” means that all the characters induced from  $C_{S_n}(u)$  have a zero valued indicator for the specified  $m$ .



TABLE 3.1. Unexpected Zero Valued Indicators

$D(S_n)$	$u$	$m$	no. of chars with $\nu_m = 0$
$S_6$	(12)(34)	3	1
$S_6$	(123)	3	1
$S_7$	(12)(34)	3	1
$S_7$	(123)	3	1
$S_8$	(12)(34)	3, 5	2
$S_8$	(12)(34)(56)(78)	5	all 7
$S_8$	(123)	3, 5	1
$S_9$	(12)(34)	3, 5	3
$S_9$	(12)(34)	5 only	3
$S_9$	(12)(34)(56)(78)	5	all 7
$S_9$	(123)	3, 5	2
$S_9$	(123)	5 only	2
$S_9$	(123)(456)	5	3
$S_9$	(123)(456)(789)	5	all 6
$S_9$	(1234)(56)	3, 5	1
$S_9$	(1234)(567)(89)	5	1
$S_9$	(12345)	3, 5	1
$S_9$	(12345)(67)(89)	5	all 2
$S_{10}$	(12)(34)	3, 5, 7, 15, 35	1
$S_{10}$	(12)(34)	3, 7	1
$S_{10}$	(12)(34)	3 only	1
$S_{10}$	(12)(34)(56)(78)	3	3
$S_{10}$	(123)	3, 5, 7, 15, 35	1
$S_{10}$	(123)	3 only	2
$S_{10}$	(123)(456)	3	1
$S_{10}$	(1234)(56)	3	1
$S_{10}$	(12345)	3	1

As we can see from the above table, it appears that the only time an unexpected zero valued indicator occurs is when  $m$  is odd. This raises the question:

**Question 3.4.4.** *Do unexpected zero valued indicators only occur when  $m$  is odd?*

We can also see that when  $m = 5$  and  $D(S_n)$  is either  $D(S_8)$  or  $D(S_9)$ , there are four occurrences of all the induced characters of specific  $C_{S_n}(u)$  having a zero valued indicator. Upon closer inspection, we find that in those instances  $\tilde{G}_5^5(u)$  is empty.

### 3.5. When $\tilde{G}_m(u)$ is not empty

In this section we consider the converse of Proposition 3.4.1. That is, we ask when it is true that if  $u$  is an even permutation of  $S_n$  or  $m$  is an even positive integer, then  $\tilde{G}_m(u) \neq \emptyset$ .

As we saw at the end of the last section, when  $u$  is an even permutation of  $S_n$  there are examples when  $\tilde{G}_5(u) = \emptyset$ , so only requiring  $u$  to be an even permutation is not sufficient. However, at least part of the converse of Proposition 3.4.1 is true. Proposition 3.5.3 below proves if  $m$  is even, then  $\tilde{G}_m(u)$  is not empty.

Interestingly enough, all our examples suggest that if  $\tilde{G}_m(u)$  is not empty, then there exists an  $h \in \tilde{G}_m(u)$  such that  $h^m = \text{id}$ . This conjecture is shown to be true for specific cases where  $u$  is an even permutation and  $m = 3$ ; see Proposition 3.5.5.

**Lemma 3.5.1.** If  $u$  is a cycle in  $S_n$  of length  $k$  and  $m$  is a positive even integer, then there exists an  $h \in S_n$  such that  $(uh)^m = h^m = \text{id}$ . Thus  $h \in \tilde{G}_m(u)$ .

*Proof.* Let  $u = (a_1 a_2 \dots a_k)$ , and

$$h = \begin{cases} (a_1 a_k)(a_2 a_{k-1}) \dots (a_{\frac{k+1}{2}}) & \text{for } k \text{ odd} \\ (a_1 a_k)(a_2 a_{k-1}) \dots (a_{\frac{k}{2}} a_{\frac{k}{2}+1}) & \text{for } k \text{ even} \end{cases}$$

Now for  $k$  odd,  $h$  consists of  $\frac{k-1}{2}$  transposition(s) and  $n - k + 1$  fixed point(s), and for  $k$  even,  $h$  consists of  $\frac{k}{2}$  transposition(s) and  $n - k$  fixed point(s).

Thus  $h^m = \text{id}$ , since  $m$  is even.

Now consider

$$uh = \begin{cases} (a_1)(a_2 a_k)(a_3 a_{k-1}) \dots (a_{\frac{k+1}{2}} a_{\frac{k+3}{2}}) & \text{for } k \text{ odd} \\ (a_1)(a_2 a_k)(a_3 a_{k-1}) \dots (a_{\frac{k}{2}+1}) & \text{for } k \text{ even} \end{cases}$$

for  $k$  odd,  $uh$  consists of  $\frac{k-1}{2}$  transposition(s) and  $n - k + 1$  fixed point(s), and for  $k$  even,  $uh$  consists of  $\frac{k}{2} - 1$  transposition(s) and  $n - k + 2$  fixed point(s).

Thus  $(uh)^m = \text{id}$ , since  $m$  is even. Therefore  $(uh)^m = h^m = \text{id}$ , and  $h \in \tilde{G}_m(u)$ .  $\square$

**Example 3.5.2.** In this example we explicitly construct the  $h \in \tilde{G}_m(u)$  as described in the proof of Lemma 3.5.1 for both an even  $k$  value (done in A) and an odd  $k$  value (done in B). Let  $m = 2s$  where  $s$  is any positive integer and let  $S_n = S_9$ .

**A)** If  $u = (3 \ 5 \ 2 \ 1 \ 4 \ 7) = (a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6)$ , so  $k = 6$ . Then  $h$  must be

$$h = (a_1 \ a_6)(a_2 \ a_5)(a_3 \ a_4) = (3 \ 7)(5 \ 4)(2 \ 1), \text{ so}$$

$$uh = (3 \ 5 \ 2 \ 1 \ 4 \ 7)(3 \ 7)(5 \ 4)(2 \ 1) = (3)(5 \ 7)(2 \ 4)(1) = (2 \ 4)(5 \ 7)$$

$$\text{thus } h^m = (h^2)^s = \text{id} = ((uh)^2)^s = (uh)^m.$$

**B)** If  $u = (3 \ 5 \ 2 \ 1 \ 4 \ 7 \ 9)$  then  $k = 7$ , and  $h$  must be

$$h = (3 \ 9)(5 \ 7)(2 \ 4)(1) \text{ or } h = (3 \ 9)(5 \ 7)(2 \ 4), \text{ so}$$

$$uh = (3 \ 5 \ 2 \ 1 \ 4 \ 7 \ 9)(3 \ 9)(5 \ 7)(2 \ 4) = (3)(5 \ 9)(2 \ 7)(1 \ 4) = (1 \ 4)(2 \ 7)(5 \ 9)$$

$$\text{thus } h^m = (h^2)^s = \text{id} = ((uh)^2)^s = (uh)^m.$$

**Proposition 3.5.3.** If  $m$  is an even positive integer, then  $\tilde{G}_m(u) \neq \emptyset$ .

Specifically, there exists an  $h \in \tilde{G}_m(u)$  such that  $h^m = \text{id}$ .

*Proof.* Let  $u \in S_n$  such that  $u = \sigma_1 \sigma_2 \dots \sigma_l$  is its cycle decomposition (so all the  $\sigma_i$  move disjoint subsets of  $1, 2, \dots, n$ ). Apply Lemma 3.5.1 to each  $\sigma_i$  to find a corresponding  $h_i$  such that  $h_i^m = (\sigma_i h_i)^m = \text{id}$ . Now  $\sigma_i$  and  $h_i$  move the same subset of  $1, 2, \dots, n$  (or  $h_i$  moves one less element than  $\sigma_i$  moves). Thus, not only are all the  $h_i$  disjoint from each other, but  $h_i$  is disjoint from  $\sigma_j$  provided  $i \neq j$ . Thus letting  $h = h_1 h_2 \dots h_l$ , we get

$$h^m = (h_1 \dots h_l)^m = h_1^m \dots h_l^m = \text{id}^l = \text{id}, \text{ and}$$

$$\begin{aligned} (uh)^m &= (\sigma_1 \sigma_2 \dots \sigma_l h_1 h_2 \dots h_l)^m = (\sigma_1 h_1 \sigma_2 h_2 \dots \sigma_l h_l)^m \\ &= (\sigma_1 h_1)^m (\sigma_2 h_2)^m \dots (\sigma_l h_l)^m = \text{id} \end{aligned}$$

Thus  $(uh)^m = h^m$  implies  $h$  is in  $\tilde{G}_m(u)$  and  $\tilde{G}_m(u) \neq \emptyset$ .  $\square$

**Example 3.5.4.** In this example we explicitly construct the  $h \in \tilde{G}_m(u)$  as described in the proof of Proposition 3.5.3. Let  $m = 2s$  where  $s \in \mathbb{N}$ , let  $S_n = S_{15}$ , and let

$$u = (3\ 5\ 2\ 1\ 4\ 7)(6\ 9)(11\ 14\ 13)(12\ 10\ 8\ 15) = \sigma_1\sigma_2\sigma_3\sigma_4. \text{ Then we construct } h_1 = (3\ 7)(5\ 4)(2\ 1), h_2 = (6\ 9), h_3 = (11\ 13)(14), \text{ and } h_4 = (12\ 15)(10\ 8), \text{ so } h = (3\ 7)(5\ 4)(2\ 1)(6\ 9)(11\ 13)(12\ 15)(10\ 8). \text{ Then}$$

$$\begin{aligned} uh &= (3)(5\ 7)(2\ 4)(1)(6)(9)(11)(14\ 13)(12)(10\ 15)(8) \\ &= (2\ 4)(5\ 7)(10\ 15)(13\ 14) \end{aligned}$$

Thus  $h^m = (h^2)^s = \text{id}^s = \text{id}$  and similarly  $(uh)^m = \text{id}$ , so  $h^m = (uh)^m$ .

**Proposition 3.5.5.** If  $u$  is an even permutation of  $S_n$ , then  $\tilde{G}_3(u)$  is not empty in the following cases:

- (1)  $u$  is a single cycle of odd length  $k = 2c + 1$ , where  $c \in \mathbb{N}$
- (2)  $u$  is the product of a transposition and a disjoint cycle of even length
- (3)  $u$  is the product of a 4-cycle and a disjoint cycle of even length at least 4

Specifically in each case, there exists an  $h$  in  $\tilde{G}_m(u)$  such that  $h^3 = \text{id}$ .

*Proof.* Consider Case (1), where  $u = (a_1 \dots a_k)$  is an even permutation. Then let  $b = \lfloor \frac{k}{4} \rfloor$ . Now construct  $h$  to be

$$h = \begin{cases} (a_k\ a_{k-1}\ a_{k-2})(a_1)(a_2\ a_{k-3}\ a_3)(a_4)(a_5\ a_{k-4}\ a_6)(a_7) \dots (a_{3b-1}) & \text{for } k \equiv 1 \pmod{4} \\ (a_k\ a_{k-1}\ a_{k-2})(a_1)(a_2\ a_{k-3}\ a_3)(a_4)(a_5\ a_{k-4}\ a_6)(a_7) \dots (a_{3b-1}\ a_{3b+1}\ a_{3b}) & \text{for } k \equiv 3 \pmod{4} \end{cases}$$

Then  $h^3 = \text{id}$  and

$$uh = \begin{cases} (a_k)(a_{k-1})(a_{k-2}\ a_1\ a_2)(a_3)(a_{k-3}\ a_4\ a_5)(a_6) \dots (a_{3b}\ a_{3b-2}\ a_{3b-1}) & \text{for } k \equiv 1 \pmod{4} \\ (a_k)(a_{k-1})(a_{k-2}\ a_1\ a_2)(a_3)(a_{k-3}\ a_4\ a_5)(a_6) \dots (a_{3b+1}) & \text{for } k \equiv 3 \pmod{4} \end{cases}$$

so  $(uh)^3 = \text{id}$ . Thus  $h^3 = (uh)^3$  and  $h \in \tilde{G}_3(u)$ .

Now consider Case (2), where  $u = \sigma_1\sigma_2 = (a_1\ a_2)(a_3 \dots a_{2r})$ , and  $r > 1$  is an integer. Notice  $u$  is an even permutation since  $\sigma_1$  is odd and  $\sigma_2$  is odd. Construct  $h$  to be

$$h = \begin{cases} (a_1a_2a_3)(a_4a_{2r}a_5)(a_6)(a_7a_{2r-1}a_8)(a_9) \dots (a_{\frac{3r}{2}+1}) & \text{for } r \text{ even} \\ (a_1a_2a_3)(a_4a_{2r}a_5)(a_6)(a_7a_{2r-1}a_8)(a_9) \dots (a_{\frac{3r-1}{2}}\ a_{\frac{3r+3}{2}}\ a_{\frac{3r+1}{2}}) & \text{for } r \text{ odd} \end{cases}$$

Then  $h^3 = \text{id}$  and

$$uh = \begin{cases} (a_1)(a_2a_4a_3)(a_5)(a_6a_7a_{2r})(a_9a_{10}a_{2r-1})(a_{11}) \dots (a_{\frac{3r}{2}}\ a_{\frac{3r}{2}+1}\ a_{\frac{3r}{2}+2}) & \text{for } r \text{ even} \\ (a_1)(a_2a_4a_3)(a_5)(a_6a_7a_{2r})(a_9a_{10}a_{2r-1})(a_{11}) \dots (a_{\frac{3r+3}{2}}) & \text{for } r \text{ odd} \end{cases}$$

so  $(uh)^3 = \text{id}$ . Thus  $h^3 = (uh)^3$  and  $h \in \tilde{G}_3(u)$ .

Now consider Case (3), where  $u = \sigma_1\sigma_2 = (a_1a_2a_3a_4)(a_5 \dots a_{2r})$ , and  $r > 3$  is an integer. Notice  $u$  is an even permutation since  $\sigma_1$  is odd and  $\sigma_2$  is odd. Construct  $h$

to be

$$h = \begin{cases} (a_2 a_1 a_5)(a_6)(a_7 a_4 a_3)(a_8 a_{2r} a_9)(a_{2r-1})(a_{10} a_{2r-2} a_{11})(a_{2r-3}) \dots (a_{r+4}) & \text{for } r \text{ even} \\ (a_2 a_1 a_5)(a_6)(a_7 a_4 a_3)(a_8 a_{2r} a_9)(a_{2r-1})(a_{10} a_{2r-2} a_{11})(a_{2r-3}) \dots & \\ \dots (a_{r+3} a_{r+5} a_{r+4}) & \text{for } r \text{ odd} \end{cases}$$

Then  $h^3 = \text{id}$  and

$$uh = \begin{cases} (a_2)(a_1 a_6 a_7)(a_4)(a_5 a_3 a_8)(a_9)(a_{10} a_{2r-1} a_{2r})(a_{11})(a_{12} a_{2r-3} a_{2r-2}) \dots & \\ \dots (a_{r+4} a_{r+5} a_{r+6}) & \text{for } r \text{ even} \\ (a_2)(a_1 a_6 a_7)(a_4)(a_5 a_3 a_8)(a_9)(a_{10} a_{2r-1} a_{2r})(a_{11})(a_{12} a_{2r-3} a_{2r-2}) \dots & \\ \dots (a_{r+5}) & \text{for } r \text{ odd} \end{cases}$$

so  $(uh)^3 = \text{id}$ . Thus  $h^3 = (uh)^3$  and  $h \in \tilde{G}_3(u)$ . □

**Example 3.5.6.** In these examples we explicitly construct the  $h \in \tilde{G}_3(u)$  as described in the proof of Proposition 3.5.5 for each of the cases. Here we let  $S_n = S_{14}$ .

**Case (1)** Let  $u = (1 \ 3 \ 7 \ 6 \ 2 \ 4 \ 5) = (a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7)$ , so  $k = 7 = 3 \pmod{4}$  and  $b = 1$ , then  $h = (a_7 \ a_6 \ a_5)(a_1)(a_2 \ a_4 \ a_3) = (5 \ 4 \ 2)(1)(3 \ 6 \ 7)$ , and  $uh = (1 \ 3 \ 7 \ 6 \ 2 \ 4 \ 5) (5 \ 4 \ 2)(1)(3 \ 6 \ 7) = (5)(4)(2 \ 1 \ 3)(6)(7) = (2 \ 1 \ 3)$ .

Thus  $(uh)^3 = h^3 = \text{id}$ .

**Case (2)** Let  $u = (1 \ 2)(3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10)$ , so  $r = 5$ . Then we construct  $h$  as

$h = (1 \ 2 \ 3)(4 \ 10 \ 5)(6)(7 \ 9 \ 8)$  which implies that  $h^3 = \text{id}$ , and

$$\begin{aligned} uh &= (1 \ 2)(3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10) (1 \ 2 \ 3)(4 \ 10 \ 5)(7 \ 9 \ 8) \\ &= (1)(2 \ 4 \ 3)(5)(6 \ 7 \ 10)(8)(9), \end{aligned}$$

so  $(uh)^3 = \text{id}$  as well.

**Case (3) A)** Let  $u = (1 \ 2 \ 3 \ 4)(5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12)$ , so  $r = 6$ . Then

$h = (2 \ 1 \ 5)(6)(7 \ 4 \ 3)(8 \ 12 \ 9)(11)(10)$  which implies that  $h^3 = \text{id}$ , and

$$\begin{aligned} uh &= (1 \ 2 \ 3 \ 4)(5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12) (2 \ 1 \ 5)(6)(7 \ 4 \ 3)(8 \ 12 \ 9)(11)(10) \\ &= (2)(1 \ 6 \ 7)(4)(5 \ 3 \ 8)(9)(10 \ 11 \ 12), \end{aligned}$$

so  $(uh)^3 = \text{id}$  as well.

**Case (3) B)** Let  $u = (1 \ 2 \ 3 \ 4)(5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14)$ , so  $r = 7$ . Then

$h = (2 \ 1 \ 5)(6)(7 \ 4 \ 3)(8 \ 14 \ 9)(13)(10 \ 12 \ 11)$  which implies that  $h^3 = \text{id}$ , and

$$\begin{aligned} uh &= (1 \ 2 \ 3 \ 4)(5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14) (2 \ 1 \ 5)(6)(7 \ 4 \ 3)(8 \ 12 \ 9)(13)(10 \ 12 \ 11) \\ &= (2)(1 \ 6 \ 7)(4)(5 \ 3 \ 8)(9)(10 \ 13 \ 14)(11)(12) \end{aligned}$$

so  $(uh)^3 = \text{id}$  as well.

We end this section with some open questions.

**Question 3.5.7.** If  $\tilde{G}_m^m(u)$  is not empty, does that imply that  $\tilde{G}_m^m(u)$  contains the identity element?

**Question 3.5.8.** If Question 3.5.7 is not true in general, is it at least true when  $m$  is an odd positive integer?

#### 4. INDICATORS OF $D(S_n)$ FOR SOME $n$

The  $m^{\text{th}}$  Frobenius-Schur indicators of a character from each irreducible I-equivalent character class of  $D(S_n)$  are presented in this section for  $n \leq 6$ . Our main result is stated in Theorem 4.0 below. It indicates that Scharf's theorem, proving all higher indicators of  $S_n$  are non-negative integers, may be plausible to extend to  $D(S_n)$ .

**Theorem 4.0** *All higher Frobenius-Schur indicators for  $D(S_n)$  for  $3 \leq n \leq 10$  are non-negative integers.*

*Proof.* See the indicator value Tables 4.1, 4.2, 4.3, 4.4, for  $n \leq 6$  and all tables in Appendix B for  $7 \leq n \leq 10$ .  $\square$

All indicators of  $D(S_n)$  were calculated in GAP by using a series of functions equivalent to the formula from Proposition 3.1.6

$$(4.0.1) \quad \nu_m(\chi_{i,j}) = \frac{1}{|C_{S_n}(u_i)|} \sum_{y \in \tilde{G}_m^m(u_i)} \Gamma_m(u_i, y) \eta_j(y)$$

where  $u_i$  is a representative of a conjugacy class in  $S_n$ ,  $\eta_j$  is an irreducible character of  $C_{S_n}(u_i)$  the centralizer of  $u_i$  in  $S_n$ , and  $\chi_{i,j}$  is the irreducible character of  $D(S_n)$  induced up from  $\eta_j$  as described in Lemma 2.2.1. Recall also that  $\tilde{G}_m^m(u)$  and  $\Gamma_m(u_i, y)$  were given in Definitions 3.1.3 and 3.1.4. Character tables containing all the characters  $\eta_j$  of a given centralizer are provided in Appendix A.

For the remainder of this paper, we use  $()$  to notate the identity element of  $S_n$ .

#### 4.1. Indicators of $D(S_3)$

In this section, we give details leading up to the use of Equation 4.0.1 to find all the irreducible I-equivalent character classes of  $D(S_3)$  and their indicator values. To use Equation 4.0.1, we first choose conjugacy class representatives of  $S_3$  so we can look at the character tables of their centralizers.

The conjugacy class representatives  $u_i$  used in these calculations and their centralizers  $C_{S_3}(u_i)$  are:

$i$	$u_i$	
1	$()$	$C_{S_3}(u_1) = S_3$ is not abelian.
2	$(1, 2)$	$C_{S_3}(u_2) = \langle (1, 2) \rangle \cong C_2$ is a cyclic abelian group.
3	$(1, 2, 3)$	$C_{S_3}(u_3) = \langle (1, 2, 3) \rangle \cong C_3$ is a cyclic abelian group.

Next we need to determine which of the higher indicators we are interested in computing. The exponent of  $S_3$  is 6, so according to Theorem 3.2.1, as long as the indicator  $\nu_m(\chi)$  is an integer when  $m$  is a divisor of 6, that is when  $m = 1, 2, 3$ , and 6, then all higher indicators will be integer valued and we will know exactly what they all are.

So we need to look at the sets  $\tilde{G}_m^m$  that we sum over, and the corresponding “coefficients”  $\Gamma_m(u, y)$  for each element  $y$  in  $\tilde{G}_m^m$ . Below we list the sets  $\tilde{G}_m(u_i)$  which give a better understanding of how each element in  $\tilde{G}_m^m(u_i)$  is found, and how the value for  $\Gamma_m(u_i, y)$  is found for each  $y$  in  $\tilde{G}_m^m(u_i)$ . Next to each set  $\tilde{G}_m(u_i)$ , we list the set  $\tilde{G}_m^m(u_i)$  with the corresponding value of  $\Gamma_m(u, y)$  paired with and preceding each element in  $\tilde{G}_m^m(u_i)$ . So if  $\tilde{G}_m^m(u_i) = \{y, \dots, z\}$  then we will display this set by writing “ $\tilde{G}_m^m(u_i)$  will consist of  $\{[\Gamma_m(u, y), y], \dots, [\Gamma_m(u, z), z]\}$ .” Thus we have:

- (1) For  $u_1 = ()$ :  
 $\tilde{G}_m(u_1) = S_3$  for all  $m$ , so  
 $\tilde{G}_1^1(u_1)$  consists of  $\{[1, ()], [3, (1, 2)], [2, (1, 2, 3)]\}$ .  
 $\tilde{G}_2^2(u_1)$  consists of  $\{[4, ()], [2, (1, 2, 3)]\}$ .  
 $\tilde{G}_3^3(u_1)$  consists of  $\{[3, ()], [3, (1, 2)]\}$ , and  
 $\tilde{G}_6^6(u_1)$  consists of  $\{[6, ()]\}$ .
- (2) For  $u_2 = (1, 2)$ :  
 $\tilde{G}_1(u_2) = \emptyset$ ,  
 $\tilde{G}_1^1(u_2) = \emptyset$ .

$$\begin{aligned}
\tilde{G}_2(u_2) &= \{(), (1, 2)\}, & \tilde{G}_2^2(u_2) &\text{consists of } \{[2, ()]\}. \\
\tilde{G}_3(u_2) &= \emptyset, & \tilde{G}_3^3(u_2) &= \emptyset. \\
\tilde{G}_6(u_2) &= S_3, & \tilde{G}_6^6(u_2) &\text{consists of } \{[6, ()]\}. \\
\textbf{(3)} \text{ For } u_3 = (1, 2, 3): & & & \\
\tilde{G}_1(u_3) &= \emptyset, & \tilde{G}_1^1(u_3) &= \emptyset. \\
\tilde{G}_2(u_3) &= \{(2, 3), (1, 2), (1, 3)\}, & \tilde{G}_2^2(u_3) &\text{consists of } \{[3, ()]\}. \\
\tilde{G}_3(u_3) &= \{(), (1, 2, 3), (1, 3, 2)\}, & \tilde{G}_3^3(u_3) &\text{consists of } \{[3, ()]\}. \\
\tilde{G}_6(u_3) &= S_3, & \tilde{G}_6^6(u_3) &\text{consists of } \{[6, ()]\}.
\end{aligned}$$

As we can see, all the  $\tilde{G}_m^m(u_2)$  are either empty or contain only the identity element, so since  $C_{S_3}(u_2)$  is abelian, Lemma 3.3.7 tells us that all characters induced from  $C_{S_3}(u_2)$  will be I-equivalent. Similarly all characters induced from  $C_{S_3}(u_3)$  will be I-equivalent.

Since we know  $\eta_{2,1}$  is I-equivalent to  $\eta_{2,2}$  by Lemma 3.3.7, and we know  $\eta_{3,1}$ ,  $\eta_{3,2}$ , and  $\eta_{3,3}$  are all three I-equivalent by Lemma 3.3.7, we do not really need the full character tables of  $C_{S_3}(u_2)$  or  $C_{S_3}(u_3)$ . We only need to know how many irreducible characters there are for each of these centralizers.

Thus using the coefficients  $\Gamma$ , the elements in  $\tilde{G}_m^m(u_i)$ , and the character tables of  $C_{S_3}(u_i)$  with Equations 4.0.1, we find:

**Proposition 4.1.1.** *The irreducible characters of  $D(S_3)$  have the following 4 distinct I-equivalent character classes:*

$$[\chi_{1.1}], [\chi_{1.2}, \chi_{3.1}, \chi_{3.2}, \chi_{3.3}], [\chi_{1.3}], [\chi_{2.1}, \chi_{2.2}].$$

The indicator values of these irreducible I-equivalent characters are displayed in Table 4.1 below.

TABLE 4.1.  $D(S_3)$  indicators: (exponent 6)

$m =$	1	2	3	6
$\nu_m(\chi_{1.1})$	0	1	0	1
$\nu_m(\chi_{1.2})$	0	1	1	2
$\nu_m(\chi_{1.3})$	1	1	1	1
$\nu_m(\chi_{2.1})$	0	1	0	3

#### 4.2. Indicators of $D(S_4)$

In this section, we give details leading up to the use of Equation 4.0.1 to find all the irreducible I-equivalent character classes of  $D(S_4)$  and their indicator values. Recall Equation 4.0.1:

$$\nu_m(\chi_{i,j}) = \frac{1}{|C_{S_n}(u_i)|} \sum_{y \in \tilde{G}_m^m(u_i)} \Gamma_m(u_i, y) \eta_j(y)$$

where  $u_i$  is a representative of a conjugacy class in  $S_n$ ,  $\eta_j$  is an irreducible character of  $C_{S_n}(u_i)$  the centralizer of  $u_i$  in  $S_n$ , and  $\chi_{i,j}$  is the irreducible character of  $D(S_n)$  induced up from  $\eta_j$  as described in Lemma 2.2.1.  $\tilde{G}_m^m(u)$  and  $\Gamma_m(u_i, y)$  were given in Definitions 3.1.3 and 3.1.4.

To begin, we first choose conjugacy class representatives of  $S_4$  so we can look at the character tables of their centralizers.

The conjugacy class representatives  $u_i$  used in these calculations and their centralizers  $C_{S_4}(u_i)$  are:

$i$	$u_i$	
1	$()$	$C_{S_4}(u_1) = S_4$ is not abelian.
2	$(1, 2)$	$C_{S_4}(u_2) = \langle (1, 2), (3, 4) \rangle \cong C_2 \times C_2$ is abelian.
3	$(1, 2)(3, 4)$	$C_{S_4}(u_3) = \langle (1, 2), (1, 3)(2, 4), (3, 4) \rangle \cong D_8$ is not abelian.
4	$(1, 2, 3)$	$C_{S_4}(u_4) = \langle (1, 2, 3) \rangle \cong C_3$ is cyclic abelian.
5	$(1, 2, 3, 4)$	$C_{S_4}(u_5) = \langle (1, 2, 3, 4) \rangle \cong C_4$ is abelian.

(1) First we consider when  $u_1 = ()$ .

The sets  $\tilde{G}_m^m(u_1)$  that we sum over, and the corresponding coefficients  $\Gamma_m(u_1, y)$  for each element  $y$  in  $\tilde{G}_m^m$  are listed below, with the corresponding value of  $\Gamma_m(u_1, y)$  preceding each element in  $\tilde{G}_m^m(u_1)$ .

Since  $\tilde{G}_m(u_1) = S_4$  for all  $m$ , we have:

$\tilde{G}_1^1(u_1)$  consists of  $\{[1, ()], [6, (1, 2)], [3, (1, 2)(3, 4)], [8, (1, 2, 3)], [6, (1, 2, 3, 4)]\}$ ;  
 $\tilde{G}_2^2(u_1)$  consists of  $\{[10, ()], [6, (1, 2)(3, 4)], [8, (1, 2, 3)]\}$ ;  
 $\tilde{G}_3^3(u_1)$  consists of  $\{[9, ()], [6, (1, 2)], [3, (1, 2)(3, 4)], [6, (1, 2, 3, 4)]\}$ ;  
 $\tilde{G}_4^4(u_1)$  consists of  $\{[16, ()], [8, (1, 2, 3)]\}$ ;  
 $\tilde{G}_6^6(u_1)$  consists of  $\{[18, ()], [6, (1, 2)(3, 4)]\}$ ;  
 $\tilde{G}_{12}^{12}(u_1)$  consists of  $\{[24, ()]\}$ .

For the rest of the  $u = u_i$ , we will list the sets  $\tilde{G}_m(u)$  which give a better understanding of how each element in  $\tilde{G}_m^m(u)$  and values for  $\Gamma_m$  were found. Then we list the sets  $\tilde{G}_m^m(u)$  with the corresponding value of  $\Gamma_m(u, y)$  preceding each element in  $\tilde{G}_m^m(u)$  in the same fashion as in Section 4.1.

In the pages that follow we will see that for  $i = 2, 4$ , and  $5$  all  $\tilde{G}_m^m(u_i)$  are either empty or only contain the identity element. So Lemma 3.3.7 applies to  $C_{S_4}(u_i)$ , for  $i = 2, 4$ , and  $5$  since each of these centralizer groups is abelian. Thus we do not really need the character tables of these centralizers, we only need to know that there are 4 irreducible characters of  $C_{S_4}(u_2)$ , 3 irreducible characters of  $C_{S_4}(u_4)$ , and 4 irreducible characters of  $C_{S_4}(u_5)$ .

(2) Now consider when  $u_2 = (1, 2)$ .

$$\begin{array}{ll}
\tilde{G}_1(u_2) = \emptyset, & \tilde{G}_1^1(u_2) = \emptyset. \\
\tilde{G}_2(u_2) = \{(), (3, 4), (1, 2), (1, 2)(3, 4)\}, & \tilde{G}_2^2(u_2) \text{ consists of } \{[4, ()]\}. \\
\tilde{G}_3(u_2) = \emptyset, & \tilde{G}_3^3(u_2) = \emptyset. \\
\tilde{G}_4(u_2) = \{(), (3, 4), (1, 2), (1, 2)(3, 4), & \tilde{G}_4^4(u_2) \text{ consists of } \{[8, ()]\}. \\
(1, 3)(2, 4), (1, 3, 2, 4), (1, 4, 2, 3), & \\
(1, 4)(2, 3)\}, & \\
\tilde{G}_6(u_2) = \{(), (3, 4), (2, 3), (2, 4), (1, 2), & \tilde{G}_6^6(u_2) \text{ consists of } \{[12, ()]\}. \\
(1, 2)(3, 4), (1, 2, 3), (1, 2, 4), (1, 3, 2), & \\
(1, 3), (1, 4, 2), (1, 4)\}, & \\
\tilde{G}_{12}(u_2) = S_4, & \tilde{G}_{12}^{12}(u_2) \text{ consists of } \{[24, ()]\}.
\end{array}$$

(3) Now consider when  $u_3 = (1, 2)(3, 4)$ .

$$\begin{array}{ll}
\tilde{G}_1(u_3) = \emptyset, & \tilde{G}_1^1(u_3) = \emptyset \\
\tilde{G}_2(u_3) = \{(), (3, 4), (1, 2), (1, 2)(3, 4), \\
\quad (1, 3)(2, 4), (1, 4)(2, 3), (1, 3, 2, 4), \\
\quad (1, 4, 2, 3)\}, & \tilde{G}_2^2(u_3) \text{ consists of } \{[6, ()], \\
& \quad [2, (1, 2)(3, 4)]\}. \\
\tilde{G}_3(u_3) = \{(2, 3, 4), (2, 4, 3), (1, 2, 3), \\
\quad (1, 2, 4), (1, 3, 2), (1, 3, 4), \\
\quad (1, 4, 2), (1, 4, 3)\}, & \tilde{G}_3^3(u_3) \text{ consists of } \{[8, ()]\}. \\
\tilde{G}_4(u_3) = \{(), (3, 4), (2, 3), (2, 4), \\
\quad (1, 2), (1, 2)(3, 4), (1, 2, 3, 4), \\
\quad (1, 2, 4, 3), (1, 3, 4, 2), (1, 3), \\
\quad (1, 3)(2, 4), (1, 3, 2, 4), (1, 4, 3, 2), \\
\quad (1, 4), (1, 4, 2, 3), (1, 4)(2, 3)\}, & \tilde{G}_4^4(u_3) \text{ consists of } \{[16, ()]\}. \\
\tilde{G}_6(u_3) = \{(), (3, 4), (2, 3, 4), (2, 4, 3), \\
\quad (1, 2), (1, 2)(3, 4), (1, 2, 3), \\
\quad (1, 2, 4), (1, 3, 2), (1, 3, 4), \\
\quad (1, 3)(2, 4), (1, 4, 2), (1, 4, 3), \\
\quad (1, 4)(2, 3), (1, 3, 2, 4), (1, 4, 2, 3)\}, & \tilde{G}_6^6(u_3) \text{ consists of } \{[14, ()], \\
& \quad [2, (1, 2)(3, 4)]\}. \\
\tilde{G}_{12}(u_3) = S_4, & \tilde{G}_{12}^{12}(u_3) \text{ consists of } \{[24, ()]\}.
\end{array}$$

(4) Now consider when  $u_4 = (1, 2, 3)$ .

$$\begin{array}{ll}
\tilde{G}_1(u_4) = \emptyset, & \tilde{G}_1^1(u_4) = \emptyset \\
\tilde{G}_2(u_4) = \{(2, 3), (1, 2), (1, 3)\}, & \tilde{G}_2^2(u_4) \text{ consists of } \{[3, ()]\}. \\
\tilde{G}_3(u_4) = \{(), (2, 4, 3), (1, 2, 3), \\
\quad (1, 3, 2), (1, 3, 4), (1, 4, 2)\}, & \tilde{G}_3^3(u_4) \text{ consists of } \{[6, ()]\}. \\
\tilde{G}_4(u_4) = \{(3, 4), (2, 3), (2, 4), (1, 2), \\
\quad (1, 2, 3, 4), (1, 2, 4, 3), (1, 3, 4, 2), \\
\quad (1, 3), (1, 3, 2, 4), (1, 4, 3, 2), \\
\quad (1, 4), (1, 4, 2, 3)\}, & \tilde{G}_4^4(u_4) \text{ consists of } \{[12, ()]\}. \\
\tilde{G}_6(u_4) = \{(), (2, 3), (2, 3, 4), (2, 4, 3), \\
\quad (1, 2), (1, 2)(3, 4), (1, 2, 3), (1, 2, 4), \\
\quad (1, 3, 2), (1, 3), (1, 3, 4), (1, 3)(2, 4), \\
\quad (1, 4, 2), (1, 4, 3), (1, 4)(2, 3)\}, & \tilde{G}_6^6(u_4) \text{ consists of } \{[15, ()]\}. \\
\tilde{G}_{12}(u_4) = S_4, & \tilde{G}_{12}^{12}(u_4) \text{ consists of } \{[24, ()]\}.
\end{array}$$



(5) Finally, consider when  $u_5 = (1, 2, 3, 4)$ .

$$\begin{aligned}
\tilde{G}_1(u_5) &= \emptyset, & \tilde{G}_1^1(u_5) &= \emptyset \\
\tilde{G}_2(u_5) &= \{(2, 4), (1, 2)(3, 4), (1, 3), & \tilde{G}_2^2(u_5) &\text{consists of } \{[4, ()]\}. \\
&\quad (1, 4)(2, 3)\}, \\
\tilde{G}_3(u_5) &= \emptyset, & \tilde{G}_3^3(u_5) &= \emptyset. \\
\tilde{G}_4(u_5) &= \{(), (2, 4), (1, 2)(3, 4), (1, 2, 3, 4), & \tilde{G}_4^4(u_5) &\text{consists of } \{[8, ()]\}. \\
&\quad (1, 3), (1, 3)(2, 4), (1, 4, 3, 2), (1, 4)(2, 3)\}, \\
\tilde{G}_6(u_5) &= \{(3, 4), (2, 3), (2, 4, 3), (2, 4), & \tilde{G}_6^6(u_5) &\text{consists of } \{[12, ()]\}. \\
&\quad (1, 2), (1, 2)(3, 4), (1, 3, 2), (1, 3), \\
&\quad (1, 4, 2), (1, 4, 3), (1, 4), (1, 4)(2, 3)\}, \\
\tilde{G}_{12}(u_5) &= S_4, & \tilde{G}_{12}^{12}(u_5) &\text{consists of } \{[24, ()]\}.
\end{aligned}$$

Thus using the coefficients  $\Gamma$ , the elements in  $\tilde{G}_m^m(u_i)$ , and the character tables of  $C_{S_4}(u_i)$  with Equations 4.0.1, we find:

**Proposition 4.2.1.** *The irreducible characters of  $D(S_4)$  have the following 7 distinct I-equivalent classes:*

$$\begin{aligned}
&[\chi_{1.1}], [\chi_{1.2}, \chi_{1.4}, \chi_{3.1}, \chi_{3.2}, \chi_{3.3}, \chi_{3.4}], [\chi_{1.3}], [\chi_{1.5}], \\
&[\chi_{2.1}, \chi_{2.2}, \chi_{2.3}, \chi_{2.4}, \chi_{5.1}, \chi_{5.2}, \chi_{5.3}, \chi_{5.4}], [\chi_{3.5}], [\chi_{4.1}, \chi_{4.2}, \chi_{4.3}].
\end{aligned}$$

The indicator values of these irreducible I-equivalent characters are displayed in Table 4.2 below.

TABLE 4.2.  $D(S_4)$  indicators: (exponent 12)

$m =$	1	2	3	4	6	12
$\nu_m(\chi_{1.1})$	0	1	0	1	1	1
$\nu_m(\chi_{1.2})$	0	1	1	2	2	3
$\nu_m(\chi_{1.3})$	0	1	1	1	2	2
$\nu_m(\chi_{1.5})$	1	1	1	1	1	1
$\nu_m(\chi_{2.1})$	0	1	0	2	3	6
$\nu_m(\chi_{3.5})$	0	1	2	4	3	6
$\nu_m(\chi_{4.1})$	0	1	2	4	5	8

#### 4.3. Indicators of $D(S_5)$

In this section, we give details leading up to the use of Equation 4.0.1 to find all the irreducible I-equivalent character classes of  $D(S_5)$  and their indicator values. Recall Equation 4.0.1:

$$\nu_m(\chi_{i.j}) = \frac{1}{|C_{S_n}(u_i)|} \sum_{y \in \tilde{G}_m^m(u_i)} \Gamma_m(u_i, y) \eta_j(y)$$

where  $u_i$  is a representative of a conjugacy class in  $S_n$ ,  $\eta_j$  is an irreducible character of  $C_{S_n}(u_i)$  the centralizer of  $u_i$  in  $S_n$ , and  $\chi_{i.j}$  is the irreducible character of  $D(S_n)$  induced up from  $\eta_j$  as described in Lemma 2.2.1.  $\tilde{G}_m^m(u)$  and  $\Gamma_m(u_i, y)$  were given in Definitions 3.1.3 and 3.1.4.

To begin, we first choose conjugacy class representatives of  $S_5$  so we can look at the character tables of their centralizers.

The conjugacy class representatives  $u_i$  used in these calculations and their centralizers  $C_{S_5}(u_i)$  are:

$i$	$u_i$	
1	$()$	$C_{S_5}(u_1) = S_5$ is not abelian.
2	$(1, 2)$	$C_{S_5}(u_2) = \langle (1, 2), (3, 5), (4, 5) \rangle \cong D_{12}$ is not abelian.
3	$(1, 2)(3, 4)$	$C_{S_5}(u_3) = \langle (1, 2), (1, 3)(2, 4), (3, 4) \rangle \cong D_8$ is not abelian.
4	$(1, 2, 3)$	$C_{S_5}(u_4) = \langle (1, 2, 3), (4, 5) \rangle \cong C_6$ is cyclic abelian.
5	$(1, 2, 3)(4, 5)$	$C_{S_5}(u_5) = \langle (1, 2, 3), (4, 5) \rangle \cong C_6$ is cyclic abelian.
6	$(1, 2, 3, 4)$	$C_{S_5}(u_6) = \langle (1, 2, 3, 4) \rangle \cong C_4$ is cyclic abelian.
7	$(1, 2, 3, 4, 5)$	$C_{S_5}(u_7) = \langle (1, 2, 3, 4, 5) \rangle \cong C_5$ is cyclic abelian.

(1) When  $u_1 = ()$ :

The sets  $\tilde{G}_m^m(u_1)$  that we sum over, and the corresponding coefficients  $\Gamma_m(u_1, y)$  for each element  $y$  in  $\tilde{G}_m^m(u_1)$  are listed below, with the corresponding value of  $\Gamma_m(u_1, y)$  preceding each element in  $\tilde{G}_m^m(u_1)$ , in the same fashion as in Section 4.1.

Since  $\tilde{G}_m(u_1) = S_5$  for all  $m$ , we have:

$\tilde{G}_1^1(u_1)$  consists of  $\{[1, ()], [10, (1, 2)], [15, (1, 2)(3, 4)], [20, (1, 2, 3)], [20, (1, 2, 3)(4, 5)], [30, (1, 2, 3, 4)], [24, (1, 2, 3, 4, 5)]\}$ ;  
 $\tilde{G}_2^2(u_1)$  consists of  $\{[26, ()], [40, (1, 2, 3)], [30, (1, 2)(3, 4)], [24, (1, 2, 3, 4, 5)]\}$ ;  
 $\tilde{G}_3^3(u_1)$  consists of  $\{[21, ()], [30, (1, 2)], [15, (1, 2)(3, 4)], [30, (1, 2, 3, 4)], [24, (1, 2, 3, 4, 5)]\}$ ;  
 $\tilde{G}_4^4(u_1)$  consists of  $\{[56, ()], [40, (1, 2, 3)], [24, (1, 2, 3, 4, 5)]\}$ ;  
 $\tilde{G}_5^5(u_1)$  consists of  $\{[25, ()], [10, (1, 2)], [15, (1, 2)(3, 4)], [20, (1, 2, 3)], [20, (1, 2, 3)(4, 5)], [30, (1, 2, 3, 4)]\}$ ;  
 $\tilde{G}_6^6(u_1)$  consists of  $\{[66, ()], [30, (1, 2)(3, 4)], [24, (1, 2, 3, 4, 5)]\}$ ;  
 $\tilde{G}_{10}^{10}(u_1)$  consists of  $\{[50, ()], [40, (1, 2, 3)], [30, (1, 2)(3, 4)]\}$ ;  
 $\tilde{G}_{12}^{12}(u_1)$  consists of  $\{[96, ()], [24, (1, 2, 3, 4, 5)]\}$ ;  
 $\tilde{G}_{15}^{15}(u_1)$  consists of  $\{[45, ()], [30, (1, 2)], [15, (1, 2)(3, 4)], [30, (1, 2, 3, 4)]\}$ ;  
 $\tilde{G}_{20}^{20}(u_1)$  consists of  $\{[80, ()], [40, (1, 2, 3)]\}$ ;  
 $\tilde{G}_{30}^{30}(u_1)$  consists of  $\{[90, ()], [30, (1, 2)(3, 4)]\}$ ;  
 $\tilde{G}_{60}^{60}(u_1)$  consists of  $\{[120, ()]\}$ .

For the rest of the  $u$ 's, we will list only the sets  $\tilde{G}_m^m(u)$  that we sum over, and the corresponding coefficients  $\Gamma_m(u, y)$  for each element  $y$  in  $\tilde{G}_m^m(u)$ , with the corresponding value of  $\Gamma_m(u, y)$  preceding each element in  $\tilde{G}_m^m(u)$  in the same fashion as above. We do not list the sets  $\tilde{G}_m(u)$  as we did in Sections 4.1 and 4.2 since these sets can be very large and too long to include in this paper.

In the pages that follow we will see that for  $i = 5, 6$ , and  $7$  all  $\tilde{G}_m^m(u_i)$  are either empty or only contain the identity element. So Lemma 3.3.7 applies to  $C_{S_5}(u_i)$ , for  $i = 5, 6$ , and  $7$  since each of these centralizer groups is abelian. Thus we do not really need the character tables of these centralizers, we only need to know that there are 6 irreducible characters of  $C_{S_5}(u_5)$ , 4 irreducible characters of  $C_{S_5}(u_6)$ , and 5 irreducible characters of  $C_{S_5}(u_7)$ .

(2) When  $u_2 = (1, 2)$ :

$\tilde{G}_1^1(u_2) = \emptyset$ ;  
 $\tilde{G}_2^2(u_2)$  consists of  $\{[8, ()], [4, (3, 4, 5)]\}$ ;

$\tilde{G}_3^3(u_2) = \emptyset$ ;  
 $\tilde{G}_4^4(u_2)$  consists of  $\{[20, ()], [4, (3, 4, 5)]\}$ ;  
 $\tilde{G}_5^5(u_2) = \emptyset$ ;  
 $\tilde{G}_6^6(u_2)$  consists of  $\{[36, ()]\}$ ;  
 $\tilde{G}_{10}^{10}(u_2)$  consists of  $\{[8, ()], [4, (3, 4, 5)]\}$ ;  
 $\tilde{G}_{12}^{12}(u_2)$  consists of  $\{[72, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_2) = \emptyset$ ;  
 $\tilde{G}_{20}^{20}(u_2)$  consists of  $\{[44, ()], [4, (3, 4, 5)]\}$ ;  
 $\tilde{G}_{30}^{30}(u_2)$  consists of  $\{[60, ()]\}$ ;  
 $\tilde{G}_{60}^{60}(u_2)$  consists of  $\{[120, ()]\}$ .

**(3)** When  $u_3 = (1, 2)(3, 4)$ :

$\tilde{G}_1^1(u_3) = \emptyset$ ;  
 $\tilde{G}_2^2(u_3)$  consists of  $\{[6, ()], [2, (1, 2)(3, 4)]\}$ ;  
 $\tilde{G}_3^3(u_3)$  consists of  $\{[8, ()]\}$ ;  
 $\tilde{G}_4^4(u_3)$  consists of  $\{[32, ()]\}$ ;  
 $\tilde{G}_5^5(u_3)$  consists of  $\{[8, ()]\}$ ;  
 $\tilde{G}_6^6(u_3)$  consists of  $\{[38, ()], [2, (1, 2)(3, 4)]\}$ ;  
 $\tilde{G}_{10}^{10}(u_3)$  consists of  $\{[30, ()], [2, (1, 2)(3, 4)]\}$ ;  
 $\tilde{G}_{12}^{12}(u_3)$  consists of  $\{[80, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_3)$  consists of  $\{[32, ()]\}$ ;  
 $\tilde{G}_{20}^{20}(u_3)$  consists of  $\{[56, ()]\}$ ;  
 $\tilde{G}_{30}^{30}(u_3)$  consists of  $\{[78, ()], [2, (1, 2)(3, 4)]\}$ ;  
 $\tilde{G}_{60}^{60}(u_3)$  consists of  $\{[120, ()]\}$ .

**(4)** When  $u_4 = (1, 2, 3)$ :

$\tilde{G}_1^1(u_4) = \emptyset$ ;  
 $\tilde{G}_2^2(u_4)$  consists of  $\{[6, ()]\}$ ;  
 $\tilde{G}_3^3(u_4)$  consists of  $\{[9, ()], [3, (4, 5)]\}$ ;  
 $\tilde{G}_4^4(u_4)$  consists of  $\{[30, ()]\}$ ;  
 $\tilde{G}_5^5(u_4)$  consists of  $\{[12, ()]\}$ ;  
 $\tilde{G}_6^6(u_4)$  consists of  $\{[36, ()]\}$ ;  
 $\tilde{G}_{10}^{10}(u_4)$  consists of  $\{[30, ()]\}$ ;  
 $\tilde{G}_{12}^{12}(u_4)$  consists of  $\{[84, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_4)$  consists of  $\{[33, ()], [3, (4, 5)]\}$ ;  
 $\tilde{G}_{20}^{20}(u_4)$  consists of  $\{[54, ()]\}$ ;  
 $\tilde{G}_{30}^{30}(u_4)$  consists of  $\{[72, ()]\}$ ;  
 $\tilde{G}_{60}^{60}(u_4)$  consists of  $\{[120, ()]\}$ .

**(5)** When  $u_5 = (1, 2, 3)(4, 5)$ :

$\tilde{G}_1^1(u_5) = \emptyset$ ;  
 $\tilde{G}_2^2(u_5)$  consists of  $\{[6, ()]\}$ ;  
 $\tilde{G}_3^3(u_5) = \emptyset$ ;  
 $\tilde{G}_4^4(u_5)$  consists of  $\{[18, ()]\}$ ;  
 $\tilde{G}_5^5(u_5) = \emptyset$ ;

$\tilde{G}_6^6(u_5)$  consists of  $\{[36, ()]\}$ ;  
 $\tilde{G}_{10}^{10}(u_5)$  consists of  $\{[18, ()]\}$ ;  
 $\tilde{G}_{12}^{12}(u_5)$  consists of  $\{[72, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_5) = \emptyset$ ;  
 $\tilde{G}_{20}^{20}(u_5)$  consists of  $\{[54, ()]\}$ ;  
 $\tilde{G}_{30}^{30}(u_5)$  consists of  $\{[60, ()]\}$ ;  
 $\tilde{G}_{60}^{60}(u_5)$  consists of  $\{[120, ()]\}$ .

(6) When  $u_6 = (1, 2, 3, 4)$ :

$\tilde{G}_1^1(u_6) = \emptyset$ ;  
 $\tilde{G}_2^2(u_6)$  consists of  $\{[4, ()]\}$ ;  
 $\tilde{G}_3^3(u_6) = \emptyset$ ;  
 $\tilde{G}_4^4(u_6)$  consists of  $\{[24, ()]\}$ ;  
 $\tilde{G}_5^5(u_6) = \emptyset$ ;  
 $\tilde{G}_6^6(u_6)$  consists of  $\{[36, ()]\}$ ;  
 $\tilde{G}_{10}^{10}(u_6)$  consists of  $\{[12, ()]\}$ ;  
 $\tilde{G}_{12}^{12}(u_6)$  consists of  $\{[72, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_6) = \emptyset$ ;  
 $\tilde{G}_{20}^{20}(u_6)$  consists of  $\{[56, ()]\}$ ;  
 $\tilde{G}_{30}^{30}(u_6)$  consists of  $\{[60, ()]\}$ ;  
 $\tilde{G}_{60}^{60}(u_6)$  consists of  $\{[120, ()]\}$ .

(7) When  $u_7 = (1, 2, 3, 4, 5)$ :

$\tilde{G}_1^1(u_7) = \emptyset$ ;  
 $\tilde{G}_2^2(u_7)$  consists of  $\{[5, ()]\}$ ;  
 $\tilde{G}_3^3(u_7)$  consists of  $\{[5, ()]\}$ ;  
 $\tilde{G}_4^4(u_7)$  consists of  $\{[30, ()]\}$ ;  
 $\tilde{G}_5^5(u_7)$  consists of  $\{[10, ()]\}$ ;  
 $\tilde{G}_6^6(u_7)$  consists of  $\{[35, ()]\}$ ;  
 $\tilde{G}_{10}^{10}(u_7)$  consists of  $\{[25, ()]\}$ ;  
 $\tilde{G}_{12}^{12}(u_7)$  consists of  $\{[80, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_7)$  consists of  $\{[35, ()]\}$ ;  
 $\tilde{G}_{20}^{20}(u_7)$  consists of  $\{[50, ()]\}$ ;  
 $\tilde{G}_{30}^{30}(u_7)$  consists of  $\{[75, ()]\}$ ;  
 $\tilde{G}_{60}^{60}(u_7)$  consists of  $\{[120, ()]\}$ .

Thus using the coefficients  $\Gamma$ , the elements in  $\tilde{G}_m^m(u_i)$ , and the character tables of  $C_{S_5}(u_i)$  with Equations 4.0.1, we find:

**Proposition 4.3.1.** *The irreducible characters of  $D(S_5)$  have the following 15 distinct  $I$ -equivalent classes:*

$$\begin{aligned}
 & [\chi_{1.1}], [\chi_{1.2}], [\chi_{1.3}, \chi_{1.5}], [\chi_{1.4}], [\chi_{1.6}], [\chi_{1.7}], \\
 & [\chi_{2.1}, \chi_{2.2}, \chi_{2.3}, \chi_{2.4}], [\chi_{2.5}, \chi_{2.6}], [\chi_{3.1}, \chi_{3.2}, \chi_{3.3}, \chi_{3.4}], [\chi_{3.5}], \\
 & [\chi_{4.1}, \chi_{4.5}, \chi_{4.6}], [\chi_{4.2}, \chi_{4.3}, \chi_{4.4}], [\chi_{5.1}, \chi_{5.2}, \chi_{5.3}, \chi_{5.4}, \chi_{5.5}, \chi_{5.6}], \\
 & [\chi_{6.1}, \chi_{6.2}, \chi_{6.3}, \chi_{6.4}], [\chi_{7.1}, \chi_{7.2}, \chi_{7.3}, \chi_{7.4}, \chi_{7.5}].
 \end{aligned}$$

The indicator values of these irreducible I-equivalent characters are displayed in Table 4.3 below.

TABLE 4.3.  $D(S_5)$  indicators: (exponent 60)

$m =$	1	2	3	4	5	6	10	12	15	20	30	60
$\nu_m(\chi_{1.1})$	0	1	0	1	0	1	1	1	0	1	1	1
$\nu_m(\chi_{1.2})$	0	1	0	2	1	2	2	3	1	3	3	4
$\nu_m(\chi_{1.3})$	0	1	1	2	1	3	2	4	2	3	4	5
$\nu_m(\chi_{1.4})$	0	1	1	3	1	3	2	5	2	4	4	6
$\nu_m(\chi_{1.6})$	0	1	1	2	1	2	2	3	2	3	3	4
$\nu_m(\chi_{1.7})$	1	1	1	1	1	1	1	1	1	1	1	1
$\nu_m(\chi_{2.1})$	0	1	0	2	0	3	1	6	0	4	5	10
$\nu_m(\chi_{2.5})$	0	1	0	3	0	6	1	12	0	7	10	20
$\nu_m(\chi_{3.1})$	0	1	1	4	1	5	4	10	4	7	10	15
$\nu_m(\chi_{3.5})$	0	1	2	8	2	9	7	20	8	14	19	30
$\nu_m(\chi_{4.1})$	0	1	2	5	2	6	5	14	6	9	12	20
$\nu_m(\chi_{4.2})$	0	1	1	5	2	6	5	14	5	9	12	20
$\nu_m(\chi_{5.1})$	0	1	0	3	0	6	3	12	0	9	10	20
$\nu_m(\chi_{6.1})$	0	1	0	6	0	9	3	18	0	14	15	30
$\nu_m(\chi_{7.1})$	0	1	1	6	2	7	5	16	7	10	15	24

#### 4.4. Indicators of $D(S_6)$

In this section, we give details leading up to the use of Equation 4.0.1 to find all the irreducible I-equivalent character classes of  $D(S_6)$  and their indicator values. Recall Equation 4.0.1:

$$\nu_m(\chi_{i.j}) = \frac{1}{|C_{S_n}(u_i)|} \sum_{y \in \tilde{G}_m^m(u_i)} \Gamma_m(u_i, y) \eta_j(y)$$

where  $u_i$  is a representative of a conjugacy class in  $S_n$ ,  $\eta_j$  is an irreducible character of  $C_{S_n}(u_i)$  the centralizer of  $u_i$  in  $S_n$ , and  $\chi_{i.j}$  is the irreducible character of  $D(S_n)$  induced up from  $\eta_j$  as described in Lemma 2.2.1.  $\tilde{G}_m^m(u)$  and  $\Gamma_m(u_i, y)$  were given in Definitions 3.1.3 and 3.1.4.

To begin, we first choose conjugacy class representatives of  $S_6$  so we can look at the character tables of their centralizers.

The conjugacy class representatives  $u_i$  used in these calculations and their centralizers  $C_{S_6}(u_i)$  are:

$i$	$u_i$	$C_{S_6}(u_i)$
1	$()$	$C_{S_6}(u_1) = S_6$ .
2	$(1, 2)$	$C_{S_6}(u_2) = \langle (1, 2), (3, 6), (4, 6), (5, 6) \rangle$ .
3	$(1, 2)(3, 4)$	$C_{S_6}(u_3) = \langle (1, 2), (1, 3)(2, 4), (3, 4), (5, 6) \rangle \cong D_8 \times C_2$ .
4	$(1, 2)(3, 4)(5, 6)$	$C_{S_6}(u_4) = \langle (1, 2), (1, 5)(2, 6), (3, 4), (3, 5)(4, 6), (5, 6) \rangle$ .
5	$(1, 2, 3)$	$C_{S_6}(u_5) = \langle (1, 2, 3), (4, 6), (5, 6) \rangle$ .
6	$(1, 2, 3)(4, 5)$	$C_{S_6}(u_6) = \langle (1, 2, 3), (4, 5) \rangle \cong C_6$ is abelian.
7	$(1, 2, 3)(4, 5, 6)$	$C_{S_6}(u_7) = \langle (1, 2, 3), (1, 4)(2, 5)(3, 6), (4, 5, 6) \rangle$ .
8	$(1, 2, 3, 4)$	$C_{S_6}(u_8) = \langle (1, 2, 3, 4), (5, 6) \rangle \cong C_4 \times C_2$ is abelian.
9	$(1, 2, 3, 4)(5, 6)$	$C_{S_6}(u_9) = \langle (1, 2, 3, 4), (5, 6) \rangle \cong C_4 \times C_2$ is abelian.
10	$(1, 2, 3, 4, 5)$	$C_{S_6}(u_{10}) = \langle (1, 2, 3, 4, 5) \rangle \cong C_5$ is abelian.
11	$(1, 2, 3, 4, 5, 6)$	$C_{S_6}(u_{11}) = \langle (1, 2, 3, 4, 5, 6) \rangle \cong C_6$ is abelian.

It is also worth noting that although  $C_{S_6}(u_2) \neq C_{S_6}(u_4)$ , they are isomorphic as groups. That is  $C_{S_6}(u_2) \cong C_{S_6}(u_4)$ . The same holds true for  $C_{S_6}(u_5) \neq C_{S_6}(u_7)$ , but  $C_{S_6}(u_5) \cong C_{S_6}(u_7)$ . And clearly  $C_{S_6}(u_8) = C_{S_6}(u_9)$ .

(1) When  $u_1 = ()$ :

The sets  $\tilde{G}_m^m(u_1)$  that we sum over, and the corresponding coefficients  $\Gamma_m(u_1, y)$  for each element  $y$  in  $\tilde{G}_m^m$  are listed below, with the corresponding value of  $\Gamma_m(u_1, y)$  preceding each element in  $\tilde{G}_m^m(u_1)$ , in the same fashion as in Section 4.1.

Since  $\tilde{G}_m(u_1) = S_6$  for all  $m$ , we have:

- $\tilde{G}_1^1(u_1)$  consists of  $\{[1, ()], [15, (1, 2)], [45, (1, 2)(3, 4)], [15, (1, 2)(3, 4)(5, 6)], [40, (1, 2, 3)], [120, (1, 2, 3)(4, 5)], [40, (1, 2, 3)(4, 5, 6)], [90, (1, 2, 3, 4)], [90, (1, 2, 3, 4)(5, 6)], [144, (1, 2, 3, 4, 5)], [120, (1, 2, 3, 4, 5, 6)]\};$
- $\tilde{G}_2^2(u_1)$  consists of  $\{[76, ()], [160, (1, 2, 3)], [160, (1, 2, 3)(4, 5, 6)], [180, (1, 2)(3, 4)], [144, (1, 2, 3, 4, 5)]\};$
- $\tilde{G}_3^3(u_1)$  consists of  $\{[81, ()], [135, (1, 2)], [45, (1, 2)(3, 4)], [135, (1, 2)(3, 4)(5, 6)], [90, (1, 2, 3, 4)], [90, (1, 2, 3, 4)(5, 6)], [144, (1, 2, 3, 4, 5)]\};$
- $\tilde{G}_4^4(u_1)$  consists of  $\{[256, ()], [160, (1, 2, 3)], [160, (1, 2, 3)(4, 5, 6)], [144, (1, 2, 3, 4, 5)]\};$
- $\tilde{G}_5^5(u_1)$  consists of  $\{[145, ()], [15, (1, 2)], [45, (1, 2)(3, 4)], [15, (1, 2)(3, 4)(5, 6)], [40, (1, 2, 3)], [120, (1, 2, 3)(4, 5)], [40, (1, 2, 3)(4, 5, 6)], [90, (1, 2, 3, 4)], [90, (1, 2, 3, 4)(5, 6)], [120, (1, 2, 3, 4, 5, 6)]\};$
- $\tilde{G}_6^6(u_1)$  consists of  $\{[396, ()], [180, (1, 2)(3, 4)], [144, (1, 2, 3, 4, 5)]\};$
- $\tilde{G}_{10}^{10}(u_1)$  consists of  $\{[220, ()], [160, (1, 2, 3)], [160, (1, 2, 3)(4, 5, 6)], [180, (1, 2)(3, 4)]\};$
- $\tilde{G}_{12}^{12}(u_1)$  consists of  $\{[576, ()], [144, (1, 2, 3, 4, 5)]\};$
- $\tilde{G}_{15}^{15}(u_1)$  consists of  $\{[225, ()], [135, (1, 2)], [45, (1, 2)(3, 4)], [135, (1, 2)(3, 4)(5, 6)], [90, (1, 2, 3, 4)], [90, (1, 2, 3, 4)(5, 6)]\};$
- $\tilde{G}_{20}^{20}(u_1)$  consists of  $\{[400, ()], [160, (1, 2, 3)], [160, (1, 2, 3)(4, 5, 6)]\};$
- $\tilde{G}_{30}^{30}(u_1)$  consists of  $\{[540, ()], [180, (1, 2)(3, 4)]\};$
- $\tilde{G}_{60}^{60}(u_1)$  consists of  $\{[720, ()]\}.$

For the rest of the  $u$ 's, we list only the sets  $\tilde{G}_m^m(u)$  that we sum over, and the corresponding coefficients  $\Gamma_m(u, y)$  for each element  $y$  in  $\tilde{G}_m^m(u)$ , with the corresponding value of  $\Gamma_m(u, y)$  preceding each element in  $\tilde{G}_m^m(u)$  in the same fashion as above. We do not list the sets  $\tilde{G}_m(u)$  as we did in Sections 4.1 and 4.2 since these sets can be very large and too long to include in this paper.

In the pages that follow we will see that for  $i = 6, 8, 9, 10$ , and  $11$  all  $\tilde{G}_m^m(u_i)$  are either empty or only contain the identity element. So Lemma 3.3.7 applies to  $C_{S_6}(u_i)$ , for  $i = 6, 8, 9, 10$ , and  $11$  since each of these centralizer groups is abelian. Thus we do not really need the character tables of these centralizers, we only need to know that there are 6 irreducible characters of  $C_{S_6}(u_6)$ , 8 irreducible characters of  $C_{S_6}(u_8)$ , 8 irreducible characters of  $C_{S_6}(u_9)$ , 5 irreducible characters of  $C_{S_6}(u_{10})$ , and 6 irreducible characters of  $C_{S_6}(u_{11})$ .

**(2)** When  $u_2 = (1, 2)$ :

$$\begin{aligned} \tilde{G}_1^1(u_2) &= \emptyset; \\ \tilde{G}_2^2(u_2) &\text{ consists of } \{[20, ()], [16, (4, 5, 6)], [12, (3, 4)(5, 6)]\}; \\ \tilde{G}_3^3(u_2) &= \emptyset; \\ \tilde{G}_4^4(u_2) &\text{ consists of } \{[80, ()], [16, (4, 5, 6)]\}; \\ \tilde{G}_5^5(u_2) &= \emptyset; \\ \tilde{G}_6^6(u_2) &\text{ consists of } \{[180, ()], [12, (3, 4)(5, 6)]\}; \\ \tilde{G}_{10}^{10}(u_2) &\text{ consists of } \{[20, ()], [16, (4, 5, 6)], [12, (3, 4)(5, 6)]\}; \\ \tilde{G}_{12}^{12}(u_2) &\text{ consists of } \{[432, ()]\}; \\ \tilde{G}_{15}^{15}(u_2) &= \emptyset; \\ \tilde{G}_{20}^{20}(u_2) &\text{ consists of } \{[176, ()], [16, (4, 5, 6)]\}; \\ \tilde{G}_{30}^{30}(u_2) &\text{ consists of } \{[372, ()], [12, (3, 4)(5, 6)]\}; \\ \tilde{G}_{60}^{60}(u_2) &\text{ consists of } \{[720, ()]\}. \end{aligned}$$

**(3)** When  $u_3 = (1, 2)(3, 4)$ :

$$\begin{aligned} \tilde{G}_1^1(u_3) &= \emptyset; \\ \tilde{G}_2^2(u_3) &\text{ consists of } \{[12, ()], [4, (1, 2)(3, 4)]\}; \\ \tilde{G}_3^3(u_3) &\text{ consists of } \{[16, ()], [8, (5, 6)], [8, (1, 2)(3, 4)(5, 6)]\}; \\ \tilde{G}_4^4(u_3) &\text{ consists of } \{[112, ()]\}; \\ \tilde{G}_5^5(u_3) &\text{ consists of } \{[64, ()]\}; \\ \tilde{G}_6^6(u_3) &\text{ consists of } \{[268, ()], [4, (1, 2)(3, 4)]\}; \\ \tilde{G}_{10}^{10}(u_3) &\text{ consists of } \{[108, ()], [4, (1, 2)(3, 4)]\}; \\ \tilde{G}_{12}^{12}(u_3) &\text{ consists of } \{[496, ()]\}; \\ \tilde{G}_{15}^{15}(u_3) &\text{ consists of } \{[144, ()], [8, (5, 6)], [8, (1, 2)(3, 4)(5, 6)]\}; \\ \tilde{G}_{20}^{20}(u_3) &\text{ consists of } \{[272, ()]\}; \\ \tilde{G}_{30}^{30}(u_3) &\text{ consists of } \{[428, ()], [4, (1, 2)(3, 4)]\}; \\ \tilde{G}_{60}^{60}(u_3) &\text{ consists of } \{[720, ()]\}. \end{aligned}$$

**(4)** When  $u_4 = (1, 2)(3, 4)(5, 6)$ :

$$\begin{aligned} \tilde{G}_1^1(u_4) &= \emptyset; \\ \tilde{G}_2^2(u_4) &\text{ consists of } \{[20, ()], [12, (3, 4)(5, 6)], [16, (1, 3, 5)(2, 4, 6)]\}; \\ \tilde{G}_3^3(u_4) &= \emptyset; \\ \tilde{G}_4^4(u_4) &\text{ consists of } \{[80, ()], [16, (1, 3, 5)(2, 4, 6)]\}; \end{aligned}$$

$\tilde{G}_5^5(u_4) = \emptyset$ ;  
 $\tilde{G}_6^6(u_4)$  consists of  $\{[180, ()], [12, (3, 4)(5, 6)]\}$ ;  
 $\tilde{G}_{10}^{10}(u_4)$  consists of  $\{[20, ()], [12, (3, 4)(5, 6)], [16, (1, 3, 5)(2, 4, 6)]\}$ ;  
 $\tilde{G}_{12}^{12}(u_4)$  consists of  $\{[432, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_4) = \emptyset$ ;  
 $\tilde{G}_{20}^{20}(u_4)$  consists of  $\{[176, ()], [16, (1, 3, 5)(2, 4, 6)]\}$ ;  
 $\tilde{G}_{30}^{30}(u_4)$  consists of  $\{[372, ()], [12, (3, 4)(5, 6)]\}$ ;  
 $\tilde{G}_{60}^{60}(u_4)$  consists of  $\{[720, ()]\}$ .

**(5)** When  $u_5 = (1, 2, 3)$ :

$\tilde{G}_1^1(u_5) = \emptyset$ ;  
 $\tilde{G}_2^2(u_5)$  consists of  $\{[12, ()], [6, (4, 5, 6)]\}$ ;  
 $\tilde{G}_3^3(u_5)$  consists of  $\{[18, ()], [18, (5, 6)]\}$ ;  
 $\tilde{G}_4^4(u_5)$  consists of  $\{[102, ()], [6, (4, 5, 6)]\}$ ;  
 $\tilde{G}_5^5(u_5)$  consists of  $\{[54, ()]\}$ ;  
 $\tilde{G}_6^6(u_5)$  consists of  $\{[252, ()]\}$ ;  
 $\tilde{G}_{10}^{10}(u_5)$  consists of  $\{[102, ()], [6, (4, 5, 6)]\}$ ;  
 $\tilde{G}_{12}^{12}(u_5)$  consists of  $\{[486, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_5)$  consists of  $\{[144, ()], [18, (5, 6)]\}$ ;  
 $\tilde{G}_{20}^{20}(u_5)$  consists of  $\{[264, ()], [6, (4, 5, 6)]\}$ ;  
 $\tilde{G}_{30}^{30}(u_5)$  consists of  $\{[414, ()]\}$ ;  
 $\tilde{G}_{60}^{60}(u_5)$  consists of  $\{[720, ()]\}$ .

**(6)** When  $u_6 = (1, 2, 3)(4, 5)$ :

$\tilde{G}_1^1(u_6) = \emptyset$ ;  
 $\tilde{G}_2^2(u_6)$  consists of  $\{[6, ()]\}$ ;  
 $\tilde{G}_3^3(u_6) = \emptyset$ ;  
 $\tilde{G}_4^4(u_6)$  consists of  $\{[96, ()]\}$ ;  
 $\tilde{G}_5^5(u_6) = \emptyset$ ;  
 $\tilde{G}_6^6(u_6)$  consists of  $\{[198, ()]\}$ ;  
 $\tilde{G}_{10}^{10}(u_6)$  consists of  $\{[30, ()]\}$ ;  
 $\tilde{G}_{12}^{12}(u_6)$  consists of  $\{[432, ()]\}$ ;  
 $\tilde{G}_{15}^{15}(u_6) = \emptyset$ ;  
 $\tilde{G}_{20}^{20}(u_6)$  consists of  $\{[192, ()]\}$ ;  
 $\tilde{G}_{30}^{30}(u_6)$  consists of  $\{[414, ()]\}$ ;  
 $\tilde{G}_{60}^{60}(u_6)$  consists of  $\{[720, ()]\}$ .

**(7)** When  $u_7 = (1, 2, 3)(4, 5, 6)$ :

$\tilde{G}_1^1(u_7) = \emptyset$ ;  $\tilde{G}_2^2(u_7)$  consists of  $\{[12, ()], [6, (1, 2, 3)(4, 6, 5)]\}$ ;  
 $\tilde{G}_3^3(u_7)$  consists of  $\{[18, ()], [18, (1, 4)(2, 5)(3, 6)]\}$ ;  
 $\tilde{G}_4^4(u_7)$  consists of  $\{[102, ()], [6, (1, 2, 3)(4, 6, 5)]\}$ ;  
 $\tilde{G}_5^5(u_7)$  consists of  $\{[54, ()]\}$ ;  
 $\tilde{G}_6^6(u_7)$  consists of  $\{[252, ()]\}$ ;  
 $\tilde{G}_{10}^{10}(u_7)$  consists of  $\{[102, ()], [6, (1, 2, 3)(4, 6, 5)]\}$ ;  
 $\tilde{G}_{12}^{12}(u_7)$  consists of  $\{[486, ()]\}$ ;



$\tilde{G}_{15}^{15}(u_7)$  consists of  $\{[144, ()], [18, (1, 4)(2, 5)(3, 6)]\}$ ;

$\tilde{G}_{20}^{20}(u_7)$  consists of  $\{[264, ()], [6, (1, 2, 3)(4, 6, 5)]\}$ ;

$\tilde{G}_{30}^{30}(u_7)$  consists of  $\{[414, ()]\}$ ;

$\tilde{G}_{60}^{60}(u_7)$  consists of  $\{[720, ()]\}$ .

**(8)** When  $u_8 = (1, 2, 3, 4)$ :

$\tilde{G}_1^1(u_8) = \emptyset$ ;

$\tilde{G}_2^2(u_8)$  consists of  $\{[8, ()]\}$ ;

$\tilde{G}_3^3(u_8) = \emptyset$ ;

$\tilde{G}_4^4(u_8)$  consists of  $\{[80, ()]\}$ ;

$\tilde{G}_5^5(u_8) = \emptyset$ ;

$\tilde{G}_6^6(u_8)$  consists of  $\{[168, ()]\}$ ;

$\tilde{G}_{10}^{10}(u_8)$  consists of  $\{[40, ()]\}$ ;

$\tilde{G}_{12}^{12}(u_8)$  consists of  $\{[432, ()]\}$ ;

$\tilde{G}_{15}^{15}(u_8) = \emptyset$ ;

$\tilde{G}_{20}^{20}(u_8)$  consists of  $\{[176, ()]\}$ ;

$\tilde{G}_{30}^{30}(u_8)$  consists of  $\{[392, ()]\}$ ;

$\tilde{G}_{60}^{60}(u_8)$  consists of  $\{[720, ()]\}$ .

**(9)** When  $u_9 = (1, 2, 3, 4)(5, 6)$ :

$\tilde{G}_1^1(u_9) = \emptyset$ ;

$\tilde{G}_2^2(u_9)$  consists of  $\{[8, ()]\}$ ;

$\tilde{G}_3^3(u_9)$  consists of  $\{[16, ()]\}$ ;

$\tilde{G}_4^4(u_9)$  consists of  $\{[80, ()]\}$ ;

$\tilde{G}_5^5(u_9)$  consists of  $\{[64, ()]\}$ ;

$\tilde{G}_6^6(u_9)$  consists of  $\{[232, ()]\}$ ;

$\tilde{G}_{10}^{10}(u_9)$  consists of  $\{[104, ()]\}$ ;

$\tilde{G}_{12}^{12}(u_9)$  consists of  $\{[496, ()]\}$ ;

$\tilde{G}_{15}^{15}(u_9)$  consists of  $\{[144, ()]\}$ ;

$\tilde{G}_{20}^{20}(u_9)$  consists of  $\{[240, ()]\}$ ;

$\tilde{G}_{30}^{30}(u_9)$  consists of  $\{[392, ()]\}$ ;

$\tilde{G}_{60}^{60}(u_9)$  consists of  $\{[720, ()]\}$ .

**(10)** When  $u_{10} = (1, 2, 3, 4, 5)$ :

$\tilde{G}_1^1(u_{10}) = \emptyset$ ;

$\tilde{G}_2^2(u_{10})$  consists of  $\{[5, ()]\}$ ;

$\tilde{G}_3^3(u_{10})$  consists of  $\{[20, ()]\}$ ;

$\tilde{G}_4^4(u_{10})$  consists of  $\{[85, ()]\}$ ;

$\tilde{G}_5^5(u_{10})$  consists of  $\{[55, ()]\}$ ;

$\tilde{G}_6^6(u_{10})$  consists of  $\{[245, ()]\}$ ;

$\tilde{G}_{10}^{10}(u_{10})$  consists of  $\{[100, ()]\}$ ;

$\tilde{G}_{12}^{12}(u_{10})$  consists of  $\{[485, ()]\}$ ;

$\tilde{G}_{15}^{15}(u_{10})$  consists of  $\{[135, ()]\}$ ;

$\tilde{G}_{20}^{20}(u_{10})$  consists of  $\{[260, ()]\}$ ;

$\tilde{G}_{30}^{30}(u_{10})$  consists of  $\{[400, ()]\}$ ;

$\tilde{G}_{60}^{60}(u_{10})$  consists of  $\{[720, ()]\}$ .

(11) When  $u_{11} = (1, 2, 3, 4, 5, 6)$ :

$\tilde{G}_1^1(u_{11}) = \emptyset$ ;

$\tilde{G}_2^2(u_{11})$  consists of  $\{[6, ()]\}$ ;

$\tilde{G}_3^3(u_{11}) = \emptyset$ ;

$\tilde{G}_4^4(u_{11})$  consists of  $\{[96, ()]\}$ ;

$\tilde{G}_5^5(u_{11}) = \emptyset$ ;

$\tilde{G}_6^6(u_{11})$  consists of  $\{[198, ()]\}$ ;

$\tilde{G}_{10}^{10}(u_{11})$  consists of  $\{[30, ()]\}$ ;

$\tilde{G}_{12}^{12}(u_{11})$  consists of  $\{[432, ()]\}$ ;

$\tilde{G}_{15}^{15}(u_{11}) = \emptyset$ ;

$\tilde{G}_{20}^{20}(u_{11})$  consists of  $\{[192, ()]\}$ ;

$\tilde{G}_{30}^{30}(u_{11})$  consists of  $\{[414, ()]\}$ ;

$\tilde{G}_{60}^{60}(u_{11})$  consists of  $\{[720, ()]\}$ .

Thus using the coefficients  $\Gamma$ , the elements in  $\tilde{G}_m^m(u_i)$ , and the character tables of  $C_{S_6}(u_i)$  with Equations 4.0.1, we find:

**Proposition 4.4.1.** *The irreducible characters of  $D(S_6)$  have the following 21 distinct I-equivalent classes:*

$$\begin{aligned} & [ \chi_{1.1} ], [ \chi_{1.2}, \chi_{1.7} ], [ \chi_{1.3} ], [ \chi_{1.4}, \chi_{1.10} ], [ \chi_{1.5}, \chi_{1.8} ], [ \chi_{1.6} ], [ \chi_{1.9} ], [ \chi_{1.11} ], \\ & [ \chi_{2.1}, \chi_{2.2}, \chi_{2.3}, \chi_{2.4}, \chi_{4.1}, \chi_{4.2}, \chi_{4.3}, \chi_{4.4} ], [ \chi_{2.5}, \chi_{2.6}, \chi_{4.5}, \chi_{4.6} ], \\ & [ \chi_{2.7}, \chi_{2.8}, \chi_{2.9}, \chi_{2.10}, \chi_{4.7}, \chi_{4.8}, \chi_{4.9}, \chi_{4.10} ], \\ & [ \chi_{3.1}, \chi_{3.6}, \chi_{3.7}, \chi_{3.8} ], [ \chi_{3.2}, \chi_{3.3}, \chi_{3.4}, \chi_{3.5} ], [ \chi_{3.9}, \chi_{3.10} ], \\ & [ \chi_{5.1}, \chi_{5.5}, \chi_{5.6}, \chi_{7.1}, \chi_{7.5}, \chi_{7.6} ], [ \chi_{5.2}, \chi_{5.3}, \chi_{5.4}, \chi_{7.2}, \chi_{7.3}, \chi_{7.4} ], \\ & [ \chi_{5.7}, \chi_{5.8}, \chi_{5.9}, \chi_{7.7}, \chi_{7.8}, \chi_{7.9} ], \\ & [ \chi_{6.1}, \chi_{6.2}, \chi_{6.3}, \chi_{6.4}, \chi_{6.5}, \chi_{6.6}, \chi_{11.1}, \chi_{11.2}, \chi_{11.3}, \chi_{11.4}, \chi_{11.5}, \chi_{11.6} ], \\ & [ \chi_{8.1}, \chi_{8.2}, \chi_{8.3}, \chi_{8.4}, \chi_{8.5}, \chi_{8.6}, \chi_{8.7}, \chi_{8.8} ], \\ & [ \chi_{9.1}, \chi_{9.2}, \chi_{9.3}, \chi_{9.4}, \chi_{9.5}, \chi_{9.6}, \chi_{9.7}, \chi_{9.8} ], \\ & [ \chi_{10.1}, \chi_{10.2}, \chi_{10.3}, \chi_{10.4}, \chi_{10.5} ]. \end{aligned}$$

The indicator values of these irreducible I-equivalent characters are displayed in the Table 4.4 below.

TABLE 4.4.  $D(S_6)$  indicators: (exponent 60)

$m =$	1	2	3	4	5	6	10	12	15	20	30	60
$\nu_m(\chi_{1.1})$	0	1	0	1	0	1	1	1	0	1	1	1
$\nu_m(\chi_{1.2})$	0	1	0	2	1	3	2	4	1	3	4	5
$\nu_m(\chi_{1.3})$	0	1	0	3	2	5	3	7	2	5	7	9
$\nu_m(\chi_{1.4})$	0	1	1	2	1	3	2	4	2	3	4	5
$\nu_m(\chi_{1.5})$	0	1	1	4	2	5	3	8	3	6	7	10
$\nu_m(\chi_{1.6})$	0	1	2	5	3	9	4	13	5	8	12	16
$\nu_m(\chi_{1.9})$	0	1	2	3	2	5	3	7	4	5	7	9
$\nu_m(\chi_{1.11})$	1	1	1	1	1	1	1	1	1	1	1	1
$\nu_m(\chi_{2.1})$	0	1	0	2	0	4	1	9	0	4	8	15
$\nu_m(\chi_{2.5})$	0	1	0	3	0	8	1	18	0	7	16	30
$\nu_m(\chi_{2.7})$	0	1	0	5	0	11	1	27	0	11	23	45
$\nu_m(\chi_{3.1})$	0	1	2	7	4	17	7	31	10	17	27	45
$\nu_m(\chi_{3.2})$	0	1	0	7	4	17	7	31	8	17	27	45
$\nu_m(\chi_{3.9})$	0	1	2	14	8	33	13	62	18	34	53	90
$\nu_m(\chi_{5.1})$	0	1	2	6	3	14	6	27	9	15	23	40
$\nu_m(\chi_{5.2})$	0	1	0	6	3	14	6	27	7	15	23	40
$\nu_m(\chi_{5.7})$	0	1	2	11	6	28	11	54	16	29	46	80
$\nu_m(\chi_{6.1})$	0	1	0	16	0	33	5	72	0	32	69	120
$\nu_m(\chi_{8.1})$	0	1	0	10	0	21	5	54	0	22	49	90
$\nu_m(\chi_{9.1})$	0	1	2	10	8	29	13	62	18	30	49	90
$\nu_m(\chi_{10.1})$	0	1	4	17	11	49	20	97	27	52	80	144

## 5. GAP FUNCTIONS AND CODE

In this section we provide the code to all of the most efficient functions we used to calculate the indicators in the Section 4, as well as the functions used to print them in a format that could be directly copied into a LaTeX file. The code for our first collection of very time consuming functions is not included. First, recall Equation 4.0.1:

$$\nu_m(\chi_{i.j}) = \frac{1}{|C_{S_n}(u_i)|} \sum_{y \in \tilde{G}_m^m(u_i)} \Gamma_m(u_i, y) \eta_j(y)$$

where  $u_i$  is a representative of a conjugacy class in  $S_n$ ,  $\eta_j$  is an irreducible character of  $C_{S_n}(u_i)$  the centralizer of  $u_i$  in  $S_n$ , and  $\chi_{i.j}$  is the irreducible character of  $D(S_n)$  induced up from  $\eta_j$  as described in Lemma 2.2.1.  $\tilde{G}_m^m(u)$  and  $\Gamma_m(u_i, y)$  were defined in Definitions 3.1.3 and 3.1.4.

This equation was coded into the multiple functions that follow in Sections 1 and 2. To compute and display all the indicator tables of  $D(S_n)$  for  $n \leq 9$  we used the last function FSLaTeXTabs (Definition 5.3.3) given in Section 3 which is built from all the functions preceeding it. In Section 4 we give a function needed for providing details seen in Sections 4.1 and 4.2, as well as a function needed to modify the size of the indicator tables. And Section 5 gives the details of how we computed the indicators of  $D(S_{10})$ .

The time needed to compute the indicator tables of  $D(S_3)$  was 2 seconds,  $D(S_4)$  was 2 seconds,  $D(S_5)$  was 3 seconds,  $D(S_6)$  was 5 seconds,  $D(S_7)$  was 20 seconds,

$D(S_8)$  was 3 minutes, and  $D(S_9)$  was a little over 1 hour. Attempting to compute the indicator tables of  $D(S_{10})$  using FSLaTeXTabs resulted in GAP error messages indicating there was not enough space to compute and store all the values at once. As an algebra,  $D(S_{10})$  has dimension  $10!^2$  or 13,168,189,440,000 which is 100 times larger than the dimension of  $D(S_9)$ . We were able to overcome these errors by breaking the code down into smaller pieces, but as a result it took about a four to five days to compute all the indicators of  $D(S_{10})$ .

Note that in GAP, all text following a  $\#$  is comment text and not part of the coding. [GAP]

### 5.1. Programming the set $\tilde{G}_m^m(u)$

There are functions in GAP that allow us to pick representatives from conjugacy classes, compute centralizer groups, find irreducible characters, and even display character tables. Thus, the first function we needed to write was how to compute the set  $\tilde{G}_m^m(u)$  and the corresponding “coefficient”  $\Gamma_m(u, y)$  for each element  $y \in \tilde{G}_m^m(u)$ .

**Definition 5.1.1.** The function  $Hmu(G, m, u)$  returns a lists of pairs  $[a, y]$ , where  $a = \Gamma_m(u, y)$  and  $y$  is an element in the set  $\tilde{G}_m^m(u)$  for the specified group  $G$ . The code for programming this function in GAP is provided below, as well as a running example of how the computation works at various steps. The code is all left justified while the running example is right justified.

```
Hmu:= function(G,m,u)
    Example: Calling Hmu( SymmetricGroup(5), 2 , (1,2) );
              in GAP means  $G = S_5$ ,  $m = 2$ , and  $u = (1, 2)$ .

    local GG, Gm, H, i, Hm, j, Cent, CC, cc, sum;
    if u = () then
        Cent:= G;
        CC:=ConjugacyClasses(G);
        Hm:=List(CC, x -> [Size(x), (Representative(x))^m]);
        # if u = (), then (uh)^m = h^m for all h,
        # so we need not check this.
    else
        GG:=EnumeratorSorted(G);
        Gm:=List(GG, x->Position(GG, x^m));
        # All elements in G are ordered and given a position number.
        # Gm stores the position number of h^m in the place of h.
        Example: GG = [(), (4, 5), (3, 4), (3, 4, 5), (3, 5, 4), (3, 5), etc ],
                  (GG)^2 = [(), (), (), (3, 5, 4), (3, 4, 5), (), etc ], so
                  Gm = [ 1, 1, 1, 5, 4, 1, etc ]

        H:=List(GG, x->[]);
        for i in [1..Size(GG)] do
            if Gm[Position(GG, u*GG[i])] = Gm[i] then
                Add(H[Gm[i]], i);
            fi;
        od;
        # (uh)^m = h^m iff the position of element (uh)^m is the
        # same as the position of the element h^m = Gm[i], so that
        # is what's checked above. All elements h that have the
```

```
# same  $h^m$  (and who satisfy  $(uh)^m = h^m$ ) are collected and
# stored (as their position in G numbers) in the position
# of  $h^m$  in G.
```

Example:  $H = [[1, 2, 3, 6, 25, 26, 27, 30], [], [], [5, 29], [4, 28], [], \text{etc}]$   
 no  $h^2 = (4, 5)$  so nothing is stored in the second place value.

```
Hm:=[];
for j in [1..Size(H)] do
  if Size(H[j]) <> 0 then
    Add( Hm, [ Size(H[j]), GG[j] ] );
  fi;
od;

# This counts how many h in G have the same  $h^m$  (that also
# satisfy  $(uh)^m = h^m$ ) and pairs this count with the
# element  $h^m$ .
```

Example:  $Hm = [[8, ()], [2, (3, 4, 5)], [2, (3, 5, 4)]]$ ,  
 so there are 8  $h$  s.t.  $h^2 = ()$ , 2  $h$  s.t.  $h^2 = (3, 4, 5)$ ,  
 and 2  $h$  s.t.  $h^2 = (3, 5, 4)$ . But  $(3, 4, 5)$  and  
 $(3, 5, 4)$  are in the same conjugacy class,  
 so these should be combined.

```
Cent:=Centralizer(G, u);
CC:=ConjugacyClasses(Cent);
fi; # if  $u = ()$ , then Cent and CC have been defined above
H:=List(Hm, x->[x[1], Position(CC, ConjugacyClass(Cent,x[2]))]);
cc:=List(H, x->x[2]);

# This takes Hm and converts the element  $h^m$  into the number
# of the position of the conjugacy class of Cent(u)
# that  $h^m$  falls into.
```

Example:  $H = [[8, 1], [2, 3], [2, 3]]$ , and  $cc = [1, 3, 3]$   
 since  $(3, 4, 5)$  &  $(3, 5, 4)$  are in the third  
 conjugacy class of  $\text{Cent}((1, 2))$ .

```
Hm:=[];
for i in DuplicateFreeList(cc) do
  sum:=0;;
  for j in Positions(cc,i) do
    sum:= sum + H[j][1];
  od;
  Add( Hm, [sum, Representative(CC[i])] );
od;
```

Example:  $\text{DuplicateFreeList}(cc) = [1, 3]$ , thus  $i = 1$  or 3  
 when  $i = 3$ ,  $\text{Positions}(cc, 3) = [2, 3]$ , giving  $j = 2$  or 3

```
return Hm;

# This returns a list of how many h in G satisfy
#  $(uh)^m = h^m$  for specific  $h^m$ , as indicated by the
# conjugacy class representative of  $h^m$ .
```

Example:  $Hm = [[8, ()], [4, (3, 4, 5)]]$

```
end;
```

## 5.2. Calculating the Indicators

The next two functions actually compute the higher indicators of  $D(G)$  by evaluating Equation 4.0.1 for specific values of  $u$  and  $m$ .

**Definition 5.2.1.**  $FSGmu(G, m, u)$  is a function which returns the  $m^{\text{th}}$  Frobenius Schur indicator of all the irreducible characters of  $D(G)$  induced from the irreducible characters of the centralizer of  $u$  in  $G$ . The code for programming this function in GAP is provided below.

```
FSGmu:= function(G,m,u)
local Centu, CharTab, Chars, CC, H, FS, sum, j, k, c, Xs, st;
if u = () then FS:=Indicator(CharacterTable(G),m);
else
  H:=Hmu(G,m,u);
  # This is a list of lists. Each sublist has two elements. The
  # first element is the number of h in G such that h^m = (uh)^m
  # for a specific h^m, and the second is the conjugacy class
  # representative in the centralizer of the element h^m.
  if H = [] then
    FS:=List(ConjugacyClasses(Centralizer(G,u)), x -> 0);
  # Since the number of irreducible characters of a finite group
  # is equal to the number of conjugacy classes of that finite
  # group, FS will have the number of conjugacy classes
  # (= number of irreducible characters) many 0's. If H is empty,
  # the indicator will be 0 for all irreducible characters.
  else
    Centu:= Centralizer(G, u);
    # This is C_G(u) the centralizer of u in G.
    CharTab:= CharacterTable(Centu);
    # This is the character table for the centralizer group.
    Chars:= Irr(CharTab);
    # This is the list of all irreducible characters and the
    # character values for each conjugacy class.
    CC:=ConjugacyClasses(CharTab);
    # This lists the conjugacy classes of Centu in the order
    # that they appear in the table.
    FS:=[];
    for j in [1..Size(Chars)] do
      # for each irreducible character find the mth
      # Frobenius-Schur indicator.
      sum:=0;
      for k in [1..Size(H)] do
        # sum over the elements h^m that come from the H.
        for c in [1..Size(CC)] do
          # find the correct column of the CharTab (aka
          # the correct ConjClass) that the h^m is in.
          if H[k][2] in CC[c] then
            sum:= sum + (H[k][1])*(Chars[j][c]);
          fi;
        end for;
      end for;
    end for;
  end if;
end if;
end function;
```

```

        od;
    od;
    FS[j]:=sum/Size(Centu);
od;
fi;
fi;
return FS;
end;

```

**Definition 5.2.2.**  $FSGu(G, mrange, u, i)$  is a function that calculates all the  $m^{\text{th}}$  Frobenius Schur indicators in  $mrange$  of all the irreducible characters of  $D(G)$  induced from the irreducible characters of the centralizer of  $u$  in  $G$ .  $mrange$  is a list of values - usually the list of divisors of the exponent of  $G$ .

The results of this function are returned in a matrix where each row corresponds to an irreducible character and each column corresponds to the  $m^{\text{th}}$  indicator values for a specified  $m$  of the corresponding irreducible character. (The number  $i$  indicates the number of the conjugacy class that  $u$  comes from). The code for programming this function in GAP is provided below.

```

FSGu:=function(G,mrange,u,i)
local FSM, MAT, r, m, Xs, j, st;
FSM:=[]; MAT:=[]; r:=1;
for m in mrange do
    FSM:= FSGmu(G,m,u);
    # Here we've found the mth Frobenius-Schur indicators
    # for all irreducible characters induced from Cent(u).
    MAT[r]:=FSM;
    # This is a row in the matrix corresponding to m.
    r:=r+1;
od;
Xs:=[];
for j in [1..Size(MAT[1])] do
    # number of columns in MAT[1] is number of irreducible
    # characters of D(G) induced from Cent(u).
    st:=String(j);
    st:=Concatenation("\\chi_{",String(i),".",st,"}");
    Xs[j]:=st;
od;
Add(MAT,Xs,1);
return TransposedMat(MAT);
end;

```

### 5.3. Condensing and Displaying Indicator Tables

Once we successfully computed all the higher indicators of  $D(G)$  we still needed to determine which irreducible characters were I-equivalent and then remove those rows from our tables. We also wanted an easy way to copy our tables of data into LaTeX so we could print them. The following functions allowed us to do just that.

**Definition 5.3.1.**  $FSMatRed(M)$  is a function that takes a Frobenius Schur indicator matrix - as given from the functions  $FSGu$  - (without the top row of  $m$  values) and

splits the first column (of all the irreducible characters) from the rest of the matrix (all the indicator values) and then reduces the matrix so there are no duplicate rows while at the same time keeping track of which rows were duplicates and thus which irreducible characters give equivalent values (that is, it finds the I-equivalent characters). The list of I-equivalent irreducible characters and reduced matrix are returned. The code for programming this function in GAP is provided below.

```
FSMatRed:=function(M)
local INDS, XS,samexs, sameinds, count, temp, i, j;
INDS:= M{[1..Size(M)]}{[2..Size(M[1])]};
    # INDS is the matrix M with the first column (the
    # characters) deleted, so INDS is just the indicator values.
XS:= M{[1..Size(M)]}{[1]};
    # XS is the column of irreducible characters taken
    # from the matrix M.
samexs:=[]; sameinds:=[];
count:=1; temp:=[];
for i in [1..Size(XS)] do
    if INDS[i] in sameinds then      # do nothing
    else      # if INDS[i] is not in the list sameinds yet, add it
        sameinds[count]:=INDS[i];
        # sameinds is the matrix of non duplicated indicator values
        # (being built row by row).
        temp:=Positions(INDS,INDS[i]);
        # temp contains the positions of the rows with identical
        # indicator values
        for j in [1..Size(temp)] do
            temp[j]:=XS[temp[j]][1];
        od;
        # temp now contains the irreducible characters
        # corresponding to each row with identical indicators.
        samexs[count]:= temp;
        # samexs is the list of I-equivalent characters
        # corresponding to the rows of sameinds.
        count:=count+1;
    fi;
od;
return [samexs, sameinds];
end;
```

**Definition 5.3.2.** *FSIndicators(num,exp)* This function returns the a list of two parts. The first part is a list of all the I-equivalent irreducible characters of  $D(S_{num})$  and the second part is the matrix of all calculated  $m^{\text{th}}$  Frobenius Schur indicators of  $D(S_{num})$  for  $m$  a divisor of  $exp$ . This matrix has no duplicate rows (or columns) and each row  $i$  corresponds to the  $i^{\text{th}}$  I-equivalency class listed in the first part. The code for programming this function in GAP is provided below.

```
FSIndicators:=function(num,exp)
local G, CCrep, factors, Mat, XS, k, FSMat, MAT, xs, samexs,
    sameinds, count, temp, i, pos, j;
```



```

factors:=DivisorsInt(exp);
G:=SymmetricGroup(num);
CCrep:= List(ConjugacyClasses(G), x -> Representative(x));
Mat:=[]; XS:=[];
for k in [1..Size(CCrep)] do          #for each CCrep do
  FSMat:= FSMatRed(FSGu(G,factors,CCrep[k],k));
  MAT:= FSMat[2];
  xs:= FSMat[1];
  # MAT is the matrix of all the irreducible characters induced by
  # CCrep[i] (in rows) with their corresponding mth indicators in
  # columns (with the corresponding text or character numbers
  # listed in xs). This separates the indicator values from the
  # character numbers. This matrix has already been reduced
  # once eliminating duplications.
  Mat:=Concatenation(Mat,MAT);
  # This stacks all the matrices on top of each other.
  XS:=Concatenation(XS, xs);
od;
samexs:=[]; sameinds:=[]; count:=1; temp:=[];
for i in [1..Size(Mat)] do
  if Mat[i] in sameinds then          # do nothing
  else                                # if Mat[i] is not in the list sameinds yet, add it
    sameinds[count]:=Mat[i];
    pos:=Positions(Mat,Mat[i]);
    temp:=[];
    for j in [1..Size(pos)] do
      temp:= Concatenation(temp,XS[pos[j]]);
    od;
    samexs[count]:= temp;
    count:=count+1;
  fi;
od;
# Ultimately sameinds is the matrix with all the Frobenius-Schur
# indicators, each row for a irreducible character and each
# column for a value of m; and samexs is a list of all
# I-equivalent irreducible characters.
return [samexs, sameinds];
end;

```

**Definition 5.3.3.** *FSLaTexTabs(num,exp)* is a function that gives a text that can be directly copied and pasted into LaTeX containing the higher Frobenius-Schur indicator table for  $D(S_{num})$ . The exponent or *exp* will provide what higher indicator values are distinct. This output specifies that there are 13 columns per table. The code for programming this function in GAP is provided below.

```

FSLaTexTabs:=function(num,exp)
local M, samexs, sameinds, factors, columns, j, numtables, k,
  range, z, l, i;
M:=FSIndicators(num, exp);

```

```

samexs:=M[1]; sameinds:=M[2];
factors:=DivisorsInt(exp);
    # Now for the display:
columns:="";
for j in [1..Size(factors)] do
    columns:=Concatenation(columns,"c");
od;
if IsInt(Size(factors)/12) then
    numtables:=Size(factors)/12;
else
    numtables:=Int(Size(factors)/12)+1;
fi;
    # The 12 means each table will have 12 values of m as
    # columns (so 13 columns total)
Print("The  $\chi^{\text{th}}$  Frobenius-Schur Indicators
of the irreducible  $\backslash n$  characters of  $D(S_{\text{num}})$ 
are: $\backslash n$ ");
for k in [1..numtables] do
    if k*12 <= Size(factors) then
        range:=[(k-1)*12+1..k*12];
    else
        range:= [(k-1)*12+1..Size(factors)];
    fi;
    if k = 1 then
        Print("\begin{table}[ht]  $\backslash n$  \caption{$D(S_{\text{num}})$:
(exponent:  $\text{\$}$ ,exp," $\text{\$}$ ) Set 1}  $\backslash n$ ");
        Print("\[  $\backslash n$ ");
    else
        Print("\begin{table}[ht]  $\backslash n$  \caption{$D(S_{\text{num}})$:
 $\text{\$m} = \text{\$}, \text{factors}[\text{range}[1]], \text{\$} \backslash \text{dots} \text{\$},
\text{factors}[\text{range}[\text{Size}(\text{range})]], \text{\$ Set 1} \text{\$} \backslash n$  \[  $\backslash n$ ");
    fi;
    Print("\begin{array}{r|",columns{range},"}  $\backslash \text{hline} \backslash n$ ");
    for j in range do
        if j = (k-1)*12+1 then Print("m = "); fi;
        Print(" & ",factors[j],"  $\backslash n$ ");
    od;
    Print("\rule[-7pt]{0pt}{20pt}  $\backslash \backslash \backslash \backslash \text{\$} \backslash \text{hline} \backslash n$ ");
    for z in [1..Size(samexs)] do
        Print("\nu_m(",samexs[z][1],")");
        for l in range do
            Print(" & ",sameinds[z][l]);
        od;
        if z = 1 or IsInt((z-1)/38)
            then Print("  $\backslash \text{rule}[0pt]{0pt}{13pt} \backslash \backslash \backslash \backslash \text{\$} \backslash n$ ");
        elif IsInt(z/38) then
            # This means each 38 rows of irreducible characters,
            # start a new table.

```

```

Print(" \\rule[-7pt]{0pt}{5pt} \\\\ \\n");
Print("\\\\hline \\n");
Print("\\\\end{array} \\n");
Print("\\\\[\\n\\label{table:S",num,".",Int(z/38),k,"}
\\n\\end{table} \\n\\n");
# end table and start a new table.
Print("\\\\begin{table}[ht] \\n \\caption{$D(S_{",num,"})$:
$m = ", factors[range[1]]," \\ldots ",
factors[range[Size(range)]],"$ Set ", 1+Int(z/38),"}
\\n\\[ \\n");
Print("\\\\begin{array}{r|",columns{range},"} \\\\hline \\n");
for j in range do
  if j = (k-1)*12+1 then Print("m = "); fi;
  Print(" & ",factors[j]," \\n");
od;
Print("\\\\rule[-7pt]{0pt}{20pt} \\\\ \\\\hline \\n");
elif z = Size(samexs)
  then Print(" \\rule[-7pt]{0pt}{5pt} \\\\ \\n");
  else Print(" \\\\ \\n");
  fi;
od;
Print("\\\\hline \\n");
Print("\\\\end{array} \\n");
Print("\\\\");
Print("\\n\\label{table:S",num,".",Int(z/38)+1,k,"}
\\n\\end{table} \\n\\n");
od;

Print("The ",Size(sameinds)," I-equivalency irreducible
character classes of $D(S_{",num,"})$ are:\\n");
Print("\\\\newline \\n");
for i in [1..Size(sameinds)] do
  if Size(samexs[i]) > 1 then
    for j in [1..Size(samexs[i])-1] do
      if j=1 then Print("[ $",samexs[i][j],",");
      elif IsInt(j/13) then
        Print(samexs[i][j],"$, \\n\\n\\hspace{.4in}$");
      else Print(samexs[i][j],",");
      fi;
      if IsInt(j/4) then Print("\\n"); fi;
    od;
    Print(samexs[i][Size(samexs[i])],"$ ],\\n");
  else
    Print("[ $",samexs[i][1],"$ ],\\n");
  fi;
  Print("\\n");
od;
return;

```

end;

#### 5.4. Additional Functions

In this section we provide the code for two more functions we found useful in when writing this dissertation. The first function steps through how  $\tilde{G}_m^m(u)$  is computed from  $\tilde{G}_m(u)$ . We used this function when writing Sections 4.1 and 4.2 when we gave the lists of elements in both of these sets. The second function became necessary when we started copying our tables of indicators and saw they were too large to fit on the page. The second function allows us to specify how many columns should be in each indicator table.

**Definition 5.4.1.** *HmuFullList*( $G, m, u$ ) This function is almost identical to *Hmu*, but it prints out the set of  $\tilde{G}_m(u)$ , and then  $\tilde{G}_m^m(u)$  in a number of different ways. The code is provided below.

```
HmuFullList:= function(G,m,u)
local GG, Gm, H, i, Hm, HM, Elem, j, Cent, CC, cc, sum;
if u = () then
  Cent:= G;
  CC:=ConjugacyClasses(G);
  Hm:=List(CC, x -> [Size(x),(Representative(x))^m]);
  Print("Since u = (), we will not print the full list of
    $\tilde{G}_m^m(u)$, instead here is \n a condensed
    list of pairs, where the \"first\" is the number of elements
    that \n when raised to the \"m,\" are equal to the
    \"second\".\n");
else
  GG:=EnumeratorSorted(G);
  Gm:=List(GG, x->Position(GG, x^m));
  H:=List(GG, x->[]);
  for i in [1..Size(GG)] do
    if Gm[Position(GG, u*GG[i])] = Gm[i] then
      Add(H[Gm[i]], i);
    fi;
  od;
  Hm:=[];HM:=[];
  for j in [1..Size(H)] do
    if Size(H[j]) <> 0 then
      Add( Hm, [ Size(H[j]), GG[j] ] );
      Elem:=List(H[j], x->GG[x]);
      Add( HM, Elem);
    fi;
  od;
  Print("$\tilde{G}_m^m(u)$ = ",Concatenation(HM)," \n");
  Print("Or if we group the elements of $\tilde{G}_m^m(u)$
    into subsets we get ",HM," \n");
  Print("Condensing this, we get a list of pairs, where the
    \"first\" is the number of \n elements that when raised to the
    \"m,\" are equal to the \"second\". \n",Hm," \n Condensing
    further to combine elements in the same conjugacy class we get
```

```

\n");
Cent:=Centralizer(G, u);
CC:=ConjugacyClasses(Cent);
fi;
H:=List(Hm, x-> [x[1], Position(CC, ConjugacyClass(Cent,x[2]))]);
cc:=List(H, x-> x[2]);
Hm:=[];
for i in DuplicateFreeList(cc) do
  sum:=0;;
  for j in Positions(cc,i) do
    sum:= sum + H[j][1];
  od;
  Add( Hm, [sum, Representative(CC[i])]);
od;
return Hm;
end;

```

**Definition 5.4.2.** *FSLaTeXTabsSpecific*( $M$ ,  $num$ ,  $exp$ ,  $tablesizes$ ) This function is very similar to *FSLaTeXTabs* in that it gives a text that can be directly copied and pasted into LaTeX containing the higher Frobenius Schur indicator table for  $D(S_{num})$ . However, there are three important differences.

First instead of calling other functions to compute the indicators, you are required to input a Matrix that contains the reduced matrix of indicators (we do this by either inputting *FSindicators*( $exp, num$ ) as  $M$  or in the case of  $S_{10}$  we can input the matrix that took multiple days to compute - see Section 5). Second the output is in LaTeX tables and not arrays, and third this function allows you to specify how many columns you want per table.

*Tablesizes* is a list of the number of  $m$  values you want in each table. The code for programming this function in GAP is provided below.

```

FSLaTeXTabsSpecific:=function(M, num, exp, tablesizes)
local factors, samexs, sameinds, numtables, columns, j, k, range,
  lowr, highr, z, l, i;
factors:=DivisorsInt(exp);
if Sum(tablesizes) <> Size(factors) then
  Print("There are ", Size(factors), " many different m values, but
    the column sizes of the tables you provided sum up to ",
    Sum(tablesizes), ". Please try again.\n");
  return;
else
samexs:=M[1]; sameinds:=M[2];
# Now for the display:
numtables:=Size(tablesizes);
columns:="";
for j in [1..Size(factors)] do
  columns:=Concatenation(columns,"c");
od;
for k in [1..numtables] do
if k = 1 then

```

```

    range:=[1..tablesizes[k]];
    lowr:=1;
else
    lowr:=1; highr:=0;
    for i in [1..k-1] do
        lowr:= lowr+tablesizes[i];
        highr:= highr+tablesizes[i];
    od;
    range:=[lowr..highr+tablesizes[k]];
fi;
if k = 1 then
    Print("\\begin{table}[ht] \n \\caption{$D(S_{",num,"})$:
      (exponent: $",exp,"$) Set 1} \n");
    Print("\\[ \n");
else
    Print("\\begin{table}[ht] \n \\caption{$D(S_{",num,"})$:
      $m = ",factors[range[1]]," \\ldots ",
        factors[range[Size(range)]], "$ Set 1} \n \\[ \n");
fi;
Print("\\begin{array}{r|",columns{range},"} \\hline \n");
for j in range do
    if j = lowr then Print("m = "); fi;
    Print(" & ",factors[j]," \n");
od;
Print("\\rule[-7pt]{0pt}{20pt} \\[\\[ \\hline \n");
for z in [1..Size(samexs)] do
Print("\\nu_m(",samexs[z][1],")");
    for l in range do
        Print(" & ",sameinds[z][l]);
    od;
    if z = 1 or IsInt((z-1)/38) then
        Print("\\rule[0pt]{0pt}{13pt} \\[\\[ \n");
    elif IsInt(z/38) then
        # This means each 38 rows of irreducible characters,
        # start a new table.
        Print("\\rule[-7pt]{0pt}{5pt} \\[\\[ \n");
        Print("\\hline \n");
        Print("\\end{array} \n");
        Print("\\[\\[\\n\\label{table:S",num,".",Int(z/38),k,"}\\n
          \\end{table} \n\\n");
    # end table and start a new table.
    Print("\\begin{table}[ht] \n \\caption{$D(S_{",num,"})$:
      $m = ", factors[range[1]]," \\ldots ",
        factors[range[Size(range)]],"$ Set ", 1+Int(z/38),"
        \n\\[ \n");
    Print("\\begin{array}{r|",columns{range},"} \\hline \n");
    for j in range do
        if j = lowr then Print("m = "); fi;

```

```

    Print(" & ",factors[j]," \n");
  od;
  Print("\rule[-7pt]{0pt}{20pt} \\\ \hline \n");
  elif z = Size(samexs) then
    Print(" \rule[-7pt]{0pt}{5pt} \\\ \n");
  else Print(" \\\ \n");
  fi;
od;
Print("\hline \n");
Print("\end{array} \n");
Print("\]");
Print("\n\\label{table:S",num,".",Int(z/38)+1,k,"}\n\\end{table}
\n\n");
od;
Print("The ",Size(sameinds)," I-equivalency irreducible character
classes of  $D(S_{",num,"})$  are:\n");
Print("\newline \n");
for i in [1..Size(sameinds)] do
  if Size(samexs[i]) > 1 then
    for j in [1..Size(samexs[i])-1] do
      if j=1 then Print("[ $",samexs[i][j]," ,");
      elif IsInt(j/13) then
        Print(samexs[i][j],"$, \n\n\\hspace{.4in}$");
      else Print(samexs[i][j]," ,");
      fi;
      if IsInt(j/4) then Print("\n"); fi;
    od;
    Print(samexs[i][Size(samexs[i])],"$ ],\n");
  else
    Print("[ $",samexs[i][1],"$, \n");
    fi;
    Print("\n");
  od;
return;
fi;
end;

```

### 5.5. Computing the Indicators of $D(S_{10})$

To compute all the indicators of  $D(S_n)$  for  $n \leq 9$  we used either FSLaTeXTabs (Definition 5.3.3) or FSLaTeXTabsSpecific (Definition 5.4.2), but when we attempted to use either one for computing the indicators of  $D(S_{10})$ , GAP produced an error message that said, “gap: cannot extend the workspace any more .” In short,  $D(S_{10})$  was too large to compute and store all the indicators at once.

Of all the previously given functions the only ones that did not give error messages when working with  $S_{10}$  were Hmu (Definition 5.1.1) and FSGmu (Definition 5.2.1). Even attempting to use FSGu (Definition 5.2.2) where we used FSGmu repeatedly to compute and store all indicators for a specific conjugacy class representative  $u$

resulted in error messages indicating there was not enough space to compute and store these all at once.

Since FSGu worked, we used the following command prompts and functions to calculate all the indicators of  $D(S_{10})$ , saved them in a single matrix M, and then used `FSLaTeXTabsSpecific(M, n, exp, [list of table sizes])` to print our tables.

(1) First we used the command prompt code displayed below to compute all the indicators for a specific conjugacy class representative  $u \in S_{10}$ , one  $m$  value at a time (where  $m$  was a divisor of  $exp = 2520$  the exponent of  $S_{10}$ ). Each computation for a specific  $m$  took between 2 and 4 minutes. The advantage of having each line print out one value of  $m$  at a time, was we were able to save the progress in case our computer lost power, over heated or had issues before the full computation was complete. When  $u = (1\ 2)$ , the code we used was:

```
>for w in DivisorsInt(9*8*7*5) do
  Print(FSGmu(SymmetricGroup(10),w,(1,2)),",\n");
od;
```

The first four lines of its output looked like this:

```
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0 ],
[ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
  1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
  1, 1 ],
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0 ],
[ 2, 2, 2, 2, 6, 6, 6, 6, 8, 8, 8, 8, 12, 12, 12, 12, 13, 13,
  13, 13, 14, 14, 14, 14, 21, 21, 21, 21, 19, 19, 28, 28,
  28, 28, 33, 33, 33, 33, 34, 34, 34, 34, 45, 45 ],
```

(2) Next we saved each of these sets of data as a matrix. Each of these matrices of indicators of irreducible characters induced from one centralizer took between 2.5 and 4.5 hours. Since there are 42 conjugacy classes of  $S_{10}$ , it took about a week to compute all 42 matrices of indicators. Once we had all the matrices, we then put each matrix through a function that added the corresponding characters to the matrix, found the I-equivalency classes, and condensed the matrix so there were no identical rows of indicators.

**Definition 5.5.1.** *FSuMatRed* This function takes the matrix of all calculated  $m^{\text{th}}$  Frobenius-Schur indicators of all the irreducible characters of  $D(G)$  induced from the irreducible characters of the centralizer of an element  $u$  in  $G$ , and it returns a list of the I-equivalency classes and the corresponding matrix of their indicator values. The number  $i$  indicates the number of the conjugacy class that  $u$  comes from. The first half of this function simply adds the list of characters to the matrix as they correspond to the indicator values. The second half of this function is identical to *FSMatRed* (Definition 5.3.1). The code for programming this function in GAP is provided below.

```
FSuMatRed:=function(MAT,i)
local Xs, j, st, M, INDS, XS, samexs, sameinds, count, temp,
```



```

pos, k;
Xs:=[];
# first we add the list of characters to this matrix
for j in [1..Size(MAT[1])] do
# number of columns in MAT[1] is number of irreducible characters
# of D(G) induced from u.
  st:=String(j);
  st:=Concatenation("\\chi_{",String(i),".",st,"}");
  Xs[j]:=st;
od;
Add(MAT,Xs,1);
M:=TransposedMat(MAT);
INDS:= M{[1..Size(M)]}{[2..Size(M[1])]};
XS:= M{[1..Size(M)]}{[1]};
samexs:=[]; sameinds:=[]; count:=1; temp:=[];
for k in [1..Size(INDS)] do
  if INDS[k] in sameinds then      #do nothing
  else      #if INDS[k] is not in the list sameinds yet, add it
    sameinds[count]:=INDS[k];
    temp:=Positions(INDS,INDS[k]);
    for j in [1..Size(temp)] do
      temp[j]:=XS[temp[j]][1];
    od;
    samexs[count]:= temp;
    count:=count+1;
  fi;
od;
return [samexs, sameinds];
end;

```

(3) After using `FSuMatRed` on all 42 of our matrices, we used the “Concatenation” command in GAP to make one long list of matrix pairs - each pair consisting of the list of I-equivalency classes and the matrix of corresponding indicator values. We then put that list through the following function to check for any mixed irreducible character I-equivalence classes.

**Definition 5.5.2.** *FSMat* This function takes a list of pairs and combines them while removing and accounting for duplications. Each pair contains the list of irreducible character I-equivalency classes of  $D(S_{10})$  as computed from one centralizer and the matrix of Frobenius Schur indicators of these classes. This function is somewhat similar to `FSIndicators` (Definition 5.3.2) The code for programming this function in GAP is provided below.

```

FSMat:=function(Mats)
local size, i, INDS, XS, samexs, sameinds, count, temp, k, pos, j;
size:=Size(Mats)/2;
if IsInt(size) then
  INDS:=[]; XS:=[];
  for i in [1.. size] do
    INDS:= Concatenation(INDS,Mats[2*i]);

```

```

XS:= Concatenation(XS,Mats[2*i-1]);
od;
samexs:=[]; sameinds:=[]; count:=1; temp:=[];
for k in [1..Size(INDS)] do
  if INDS[k] in sameinds then      #do nothing
  else      #if INDS[k] is not in the list sameinds yet, add it
    sameinds[count]:=INDS[k];
    pos:=Positions(INDS,INDS[k]);
    temp:=[];
    for j in [1..Size(pos)] do
      temp:= Concatenation(temp,XS[pos[j]]);
    od;
    samexs[count]:= temp;
    count:=count+1;
  fi;
od;
return [samexs, sameinds];
else return "bad input";
fi;
end;

```

(4) The result from FSMat gave us the matrix M, consisting of first the list of all the I-equivalency classes of  $D(S_{10})$  and second the matrix of corresponding indicator values, that we needed to run FSLaTeXTabsSpecific (Definition 5.4.2).

Attempting to run any of our functions using  $S_{11}$  resulted in errors from GAP.

## APPENDIX A. CHARACTER TABLES

This appendix provides all character tables used in calculating all the higher indicators of  $D(S_3)$  to  $D(S_6)$ . Although the characters of these centralizers are well known, the order in which they are displayed in tables varies from source to source. Therefore, we include them here to make our notation of the characters of  $D(S_n)$  clear. Note that in the tables that follow, for  $n = 3, 4$  and  $5$ ,  $E(n)$  denotes a primitive  $n^{\text{th}}$  root of 1.

### A.1. Character Tables of the Centralizers of $S_3$

The character tables of the centralizers in  $S_3$  of conjugacy class representatives of  $S_3$  are given below. These tables were used to calculate the indicators of  $D(S_3)$  recorded in Section 4.1.

TABLE A.1. Character tables of Centralizers of  $S_3$

$C_{S_3}(( )) \cong S_3 :$				$C_{S_3}((1, 2)) \cong C_2 :$			$C_{S_3}((1, 2, 3)) \cong C_3 :$			
	( )	(1, 2)	(1, 2, 3)		( )	(1, 2)		( )	(1, 2, 3)	(1, 3, 2)
$\eta_{1.1}$	1	-1	1	$\eta_{2.1}$	1	-1	$\eta_{3.1}$	1	1	1
$\eta_{1.2}$	2	0	-1	$\eta_{2.2}$	1	1	$\eta_{3.2}$	1	$E(3)$	$E(3)^2$
$\eta_{1.3}$	1	1	1				$\eta_{3.3}$	1	$E(3)^2$	$E(3)$

**A.2. Character Tables of the Centralizers of  $S_4$** 

The character tables of the centralizers in  $S_4$  of conjugacy class representatives of  $S_4$  are given below. These tables were used to calculate the indicators of  $D(S_4)$  recorded in Section 4.2.

TABLE A.2. Character table of  $C_{S_4}(( )) \cong S_4$ 

	$()$	$(1, 2)$	$(1, 2)(3, 4)$	$(1, 2, 3)$	$(1, 2, 3, 4)$
$\eta_{1.1}$	1	-1	1	1	-1
$\eta_{1.2}$	3	-1	-1	0	1
$\eta_{1.3}$	2	0	2	-1	0
$\eta_{1.4}$	3	1	-1	0	-1
$\eta_{1.5}$	1	1	1	1	1

TABLE A.3. Character table of  $C_{S_4}((1, 2)) \cong \langle (1, 2), (3, 4) \rangle \cong C_2 \times C_2$ 

	$()$	$(3, 4)$	$(1, 2)$	$(1, 2)(3, 4)$
$\eta_{2.1}$	1	1	1	1
$\eta_{2.2}$	1	-1	-1	1
$\eta_{2.3}$	1	-1	1	-1
$\eta_{2.4}$	1	1	-1	-1

TABLE A.4. Character table of  $C_{S_4}((1, 2)(3, 4)) \cong \langle (1, 2), (1, 3)(2, 4), (3, 4) \rangle \cong D_8$ 

	$()$	$(3, 4)$	$(1, 2)(3, 4)$	$(1, 3)(2, 4)$	$(1, 3, 2, 4)$
$\eta_{3.1}$	1	1	1	1	1
$\eta_{3.2}$	1	-1	1	-1	1
$\eta_{3.3}$	1	-1	1	1	-1
$\eta_{3.4}$	1	1	1	-1	-1
$\eta_{3.5}$	2	0	-2	0	0

TABLE A.5. Character table of  $C_{S_4}((1, 2, 3)) \cong \langle (1, 2, 3) \rangle \cong C_3$ 

	$()$	$(1, 2, 3)$	$(1, 3, 2)$
$\eta_{4.1}$	1	1	1
$\eta_{4.2}$	1	$E(3)$	$E(3)^2$
$\eta_{4.3}$	1	$E(3)^2$	$E(3)$

TABLE A.6. Character table of  $C_{S_4}((1, 2, 3, 4)) \cong \langle (1, 2, 3, 4) \rangle \cong C_4$ 

	()	(1, 2, 3, 4)	(1, 3)(2, 4)	(1, 4, 3, 2)
$\eta_{5.1}$	1	1	1	1
$\eta_{5.2}$	1	-1	1	-1
$\eta_{5.3}$	1	$E(4)$	-1	$-E(4)$
$\eta_{5.4}$	1	$-E(4)$	-1	$E(4)$

**A.3. Character Tables of the Centralizers of  $S_5$** 

The character tables of the centralizers in  $S_5$  of conjugacy class representatives of  $S_5$  are given below. These tables were used to calculate the indicators of  $D(S_5)$  recorded in Section 4.3.

TABLE A.7. Character table of  $C_{S_5}(( )) \cong S_5$ 

	()	(1, 2)	(1, 2)(3, 4)	(1, 2, 3)	(1, 2, 3)(4, 5)	(1, 2, 3, 4)	(1, 2, 3, 4, 5)
$\eta_{1.1}$	1	-1	1	1	-1	-1	1
$\eta_{1.2}$	4	-2	0	1	1	0	-1
$\eta_{1.3}$	5	-1	1	-1	-1	1	0
$\eta_{1.4}$	6	0	-2	0	0	0	1
$\eta_{1.5}$	5	1	1	-1	1	-1	0
$\eta_{1.6}$	4	2	0	1	-1	0	-1
$\eta_{1.7}$	1	1	1	1	1	1	1

TABLE A.8. Character table of  $C_{S_5}((1, 2)) \cong \langle (1, 2), (3, 5), (4, 5) \rangle \cong D_{12}$ 

	()	(4, 5)	(3, 4, 5)	(1, 2)	(1, 2)(4, 5)	(1, 2)(3, 4, 5)
$\eta_{2.1}$	1	1	1	1	1	1
$\eta_{2.2}$	1	-1	1	-1	1	-1
$\eta_{2.3}$	1	-1	1	1	-1	1
$\eta_{2.4}$	1	1	1	-1	-1	-1
$\eta_{2.5}$	2	0	-1	-2	0	1
$\eta_{2.6}$	2	0	-1	2	0	-1

TABLE A.9. Character table of  $C_{S_5}((1, 2)(3, 4)) \cong \langle (1, 2), (1, 3)(2, 4), (3, 4) \rangle \cong D_8$ 

	()	(3, 4)	(1, 2)(3, 4)	(1, 3)(2, 4)	(1, 3, 2, 4)
$\eta_{3.1}$	1	1	1	1	1
$\eta_{3.2}$	1	-1	1	-1	1
$\eta_{3.3}$	1	-1	1	1	-1
$\eta_{3.4}$	1	1	1	-1	-1
$\eta_{3.5}$	2	0	-2	0	0

TABLE A.10. Character table of  $C_{S_5}((1, 2, 3)) \cong \langle (1, 2, 3), (4, 5) \rangle \cong C_6$ 

	()	(4, 5)	(1, 2, 3)	(1, 2, 3)(4, 5)	(1, 3, 2)	(1, 3, 2)(4, 5)
$\eta_{4.1}$	1	1	1	1	1	1
$\eta_{4.2}$	1	-1	1	-1	1	-1
$\eta_{4.3}$	1	-1	$E(3)^2$	$-E(3)^2$	$E(3)$	$-E(3)$
$\eta_{4.4}$	1	-1	$E(3)$	$-E(3)$	$E(3)^2$	$-E(3)^2$
$\eta_{4.5}$	1	1	$E(3)^2$	$E(3)^2$	$E(3)$	$E(3)$
$\eta_{4.6}$	1	1	$E(3)$	$E(3)$	$E(3)^2$	$E(3)^2$

The character table of  $C_{S_5}((1, 2, 3)(4, 5)) \cong \langle (1, 2, 3), (4, 5) \rangle \cong C_6$  is identical to the table given just above, with the only exception being the characters are indexed by 5. $j$ . Each of the 6 irreducible characters  $\eta_{5,j}$  has the same character values as the corresponding irreducible characters  $\eta_{4,j}$  listed in the table immediately above.

TABLE A.11. Character table of  $C_{S_5}((1, 2, 3, 4)) \cong \langle (1, 2, 3, 4) \rangle \cong C_4$ 

	()	(1, 2, 3, 4)	(1, 3)(2, 4)	(1, 4, 3, 2)
$\eta_{6.1}$	1	1	1	1
$\eta_{6.2}$	1	-1	1	-1
$\eta_{6.3}$	1	$E(4)$	-1	$-E(4)$
$\eta_{6.4}$	1	$-E(4)$	-1	$E(4)$

TABLE A.12. Character table of  $C_{S_5}((1, 2, 3, 4, 5)) \cong \langle (1, 2, 3, 4, 5) \rangle \cong C_5$ 

	()	(1, 2, 3, 4, 5)	(1, 3, 5, 2, 4)	(1, 4, 2, 5, 3)	(1, 5, 4, 3, 2)
$\eta_{7.1}$	1	1	1	1	1
$\eta_{7.2}$	1	$E(5)$	$E(5)^2$	$E(5)^3$	$E(5)^4$
$\eta_{7.3}$	1	$E(5)^2$	$E(5)^4$	$E(5)$	$E(5)^3$
$\eta_{7.4}$	1	$E(5)^3$	$E(5)$	$E(5)^4$	$E(5)^2$
$\eta_{7.5}$	1	$E(5)^4$	$E(5)^3$	$E(5)^2$	$E(5)$

#### A.4. Character Tables of the Centralizers of $S_6$

The character tables of the centralizers in  $S_6$  of conjugacy class representatives of  $S_6$  are given below. These tables were used to calculate the indicators of  $D(S_6)$  recorded in Section 4.4.

TABLE A.13. Character table of  $C_{S_6}(( )) \cong S_6$ 

	1a	2a	2b	2c	3a	6a	3b	4a	4b	5a	6b
$\eta_{1.1}$	1	-1	1	-1	1	-1	1	-1	1	1	-1
$\eta_{1.2}$	5	-3	1	1	2	0	-1	-1	-1	0	1
$\eta_{1.3}$	9	-3	1	-3	0	0	0	1	1	-1	0
$\eta_{1.4}$	5	-1	1	3	-1	-1	2	1	-1	0	0
$\eta_{1.5}$	10	-2	-2	2	1	1	1	0	0	0	-1
$\eta_{1.6}$	16	0	0	0	-2	0	-2	0	0	1	0
$\eta_{1.7}$	5	1	1	-3	-1	1	2	-1	-1	0	0
$\eta_{1.8}$	10	2	-2	-2	1	-1	1	0	0	0	1
$\eta_{1.9}$	9	3	1	3	0	0	0	-1	1	-1	0
$\eta_{1.10}$	5	3	1	-1	2	0	-1	1	-1	0	-1
$\eta_{1.11}$	1	1	1	1	1	1	1	1	1	1	1

Here 1a =  $()$ , 2a =  $(1,2)$ , 2b =  $(1,2)(3,4)$ , 2c =  $(1,2)(3,4)(5,6)$ , 3a =  $(1,2,3)$ , 6a =  $(1,2,3)(4,5)$ , 3b =  $(1,2,3)(4,5,6)$ , 4a =  $(1,2,3,4)$ , 4b =  $(1,2,3,4)(5,6)$ , 5a =  $(1,2,3,4,5)$ , 6b =  $(1,2,3,4,5,6)$ .

TABLE A.14. Character table of  $C_{S_6}((1,2)) \cong \langle (1,2), (3,6), (4,6), (5,6) \rangle$ 

	1a	2a	3a	2b	4a	2c	2d	6a	2e	4b
$\eta_{2.1}$	1	1	1	1	1	1	1	1	1	1
$\eta_{2.2}$	1	-1	1	1	-1	-1	1	-1	-1	1
$\eta_{2.3}$	1	-1	1	1	-1	1	-1	1	1	-1
$\eta_{2.4}$	1	1	1	1	1	-1	-1	-1	-1	-1
$\eta_{2.5}$	2	0	-1	2	0	-2	0	1	-2	0
$\eta_{2.6}$	2	0	-1	2	0	2	0	-1	2	0
$\eta_{2.7}$	3	-1	0	-1	1	-3	1	0	1	-1
$\eta_{2.8}$	3	-1	0	-1	1	3	-1	0	-1	1
$\eta_{2.9}$	3	1	0	-1	-1	-3	-1	0	1	1
$\eta_{2.10}$	3	1	0	-1	-1	3	1	0	-1	-1

Here 1a =  $()$ , 2a =  $(5,6)$ , 3a =  $(4,5,6)$ , 2b =  $(3,4)(5,6)$ , 4a =  $(3,4,5,6)$ , 2c =  $(1,2)$ , 2d =  $(1,2)(5,6)$ , 6a =  $(1,2)(4,5,6)$ , 2e =  $(1,2)(3,4)(5,6)$ , 4b =  $(1,2)(3,4,5,6)$ .

TABLE A.15. Character table of  $C_{S_6}((1,2)(3,4)) \cong \langle (1,2), (1,3)(2,4), (3,4), (5,6) \rangle \cong D_8 \times C_2$

	1a	2a	2b	2c	2d	2e	2f	2g	4a	4b
$\eta_{3.1}$	1	1	1	1	1	1	1	1	1	1
$\eta_{3.2}$	1	-1	-1	1	1	-1	-1	1	1	-1
$\eta_{3.3}$	1	-1	-1	1	1	-1	1	-1	-1	1
$\eta_{3.4}$	1	-1	1	-1	1	-1	-1	1	-1	1
$\eta_{3.5}$	1	-1	1	-1	1	-1	1	-1	1	-1
$\eta_{3.6}$	1	1	-1	-1	1	1	-1	-1	1	1
$\eta_{3.7}$	1	1	-1	-1	1	1	1	1	-1	-1
$\eta_{3.8}$	1	1	1	1	1	1	-1	-1	-1	-1
$\eta_{3.9}$	2	2	0	0	-2	-2	0	0	0	0
$\eta_{3.10}$	2	-2	0	0	-2	2	0	0	0	0

Here 1a = (), 2a = (5,6), 2b = (3,4), 2c = (3,4)(5,6), 2d = (1,2)(3,4), 2e = (1,2)(3,4)(5,6), 2f = (1,3)(2,4), 2g = (1,3)(2,4)(5,6), 4a = (1,3,2,4), 4b = (1,3,2,4)(5,6).

TABLE A.16. Character table of  $C_{S_6}((1,2)(3,4)(5,6)) \cong \langle (1,2), (1,5)(2,6), (3,4), (3,5)(4,6), (5,6) \rangle$

	1a	2a	2b	2c	4a	2d	2e	4b	3a	6a
$\eta_{4.1}$	1	1	1	1	1	1	1	1	1	1
$\eta_{4.2}$	1	-1	1	-1	1	-1	1	-1	1	-1
$\eta_{4.3}$	1	-1	1	1	-1	-1	-1	1	1	-1
$\eta_{4.4}$	1	1	1	-1	-1	1	-1	-1	1	1
$\eta_{4.5}$	2	-2	2	0	0	-2	0	0	-1	1
$\eta_{4.6}$	2	2	2	0	0	2	0	0	-1	-1
$\eta_{4.7}$	3	-1	-1	-1	1	3	-1	1	0	0
$\eta_{4.8}$	3	-1	-1	1	-1	3	1	-1	0	0
$\eta_{4.9}$	3	1	-1	-1	-1	-3	1	1	0	0
$\eta_{4.10}$	3	1	-1	1	1	-3	-1	-1	0	0

Here 1a = (), 2a = (5,6), 2b = (3,4)(5,6), 2c = (3,5)(4,6), 4a = (3,5,4,6), 2d = (1,2)(3,4)(5,6), 2e = (1,2)(3,5)(4,6), 4b = (1,2)(3,5,4,6), 3a = (1,3,5)(2,4,6), 6a = (1,3,5,2,4,6).

TABLE A.17. Character table of  $C_{S_6}((1, 2, 3)) \cong \langle (1, 2, 3), (4, 6), (5, 6) \rangle$ 

	1a	2a	3a	3b	6a	3c	3d	6b	3e
$\eta_{5.1}$	1	1	1	1	1	1	1	1	1
$\eta_{5.2}$	1	-1	1	1	-1	1	1	-1	1
$\eta_{5.3}$	1	-1	1	$E(3)^2$	$-E(3)^2$	$E(3)^2$	$E(3)$	$-E(3)$	$E(3)$
$\eta_{5.4}$	1	-1	1	$E(3)$	$-E(3)$	$E(3)$	$E(3)^2$	$-E(3)^2$	$E(3)^2$
$\eta_{5.5}$	1	1	1	$E(3)^2$	$E(3)^2$	$E(3)^2$	$E(3)$	$E(3)$	$E(3)$
$\eta_{5.6}$	1	1	1	$E(3)$	$E(3)$	$E(3)$	$E(3)^2$	$E(3)^2$	$E(3)^2$
$\eta_{5.7}$	2	0	-1	2	0	-1	2	0	-1
$\eta_{5.8}$	2	0	-1	$2 * E(3)$	0	$-E(3)$	$2 * E(3)^2$	0	$-E(3)^2$
$\eta_{5.9}$	2	0	-1	$2 * E(3)^2$	0	$-E(3)^2$	$2 * E(3)$	0	$-E(3)$

Here 1a = (), 2a = (5,6), 3a = (4,5,6), 3b = (1,2,3), 6a = (1,2,3)(5,6), 3c = (1,2,3)(4,5,6), 3d = (1,3,2), 6b = (1,3,2)(5,6), 3e = (1,3,2)(4,5,6).

TABLE A.18. Character table of  $C_{S_6}((1, 2, 3)(4, 5)) \cong \langle (1, 2, 3), (4, 5) \rangle \cong C_6$ 

	1a	2a	3a	6a	3b	6b
$\eta_{6.1}$	1	1	1	1	1	1
$\eta_{6.2}$	1	-1	1	-1	1	-1
$\eta_{6.3}$	1	-1	$E(3)^2$	$-E(3)^2$	$E(3)$	$-E(3)$
$\eta_{6.4}$	1	-1	$E(3)$	$-E(3)$	$E(3)^2$	$-E(3)^2$
$\eta_{6.5}$	1	1	$E(3)^2$	$E(3)^2$	$E(3)$	$E(3)$
$\eta_{6.6}$	1	1	$E(3)$	$E(3)$	$E(3)^2$	$E(3)^2$

Here 1a = (), 2a = (4,5), 3a = (1,2,3), 6a = (1,2,3)(4,5), 3b = (1,3,2), 6b = (1,3,2)(4,5).

TABLE A.19. Character table of  $C_{S_6}((1, 2, 3)(4, 5, 6)) \cong \langle (1, 2, 3), (1, 4)(2, 5)(3, 6), (4, 5, 6) \rangle$ 

	1a	3a	3b	3c	3d	3e	2a	6a	6b
$\eta_{7.1}$	1	1	1	1	1	1	1	1	1
$\eta_{7.2}$	1	1	1	1	1	1	-1	-1	-1
$\eta_{7.3}$	1	$E(3)^2$	$E(3)$	$E(3)$	1	$E(3)^2$	-1	$-E(3)^2$	$-E(3)$
$\eta_{7.4}$	1	$E(3)$	$E(3)^2$	$E(3)^2$	1	$E(3)$	-1	$-E(3)$	$-E(3)^2$
$\eta_{7.5}$	1	$E(3)^2$	$E(3)$	$E(3)$	1	$E(3)^2$	1	$E(3)^2$	$E(3)$
$\eta_{7.6}$	1	$E(3)$	$E(3)^2$	$E(3)^2$	1	$E(3)$	1	$E(3)$	$E(3)^2$
$\eta_{7.7}$	2	-1	-1	2	-1	2	0	0	0
$\eta_{7.8}$	2	$-E(3)$	$-E(3)^2$	$2 * E(3)^2$	-1	$2 * E(3)$	0	0	0
$\eta_{7.9}$	2	$-E(3)^2$	$-E(3)$	$2 * E(3)$	-1	$2 * E(3)^2$	0	0	0

Here 1a = (), 3a = (4,5,6), 3b = (4,6,5), 3c = (1,2,3)(4,5,6), 3d = (1,2,3)(4,6,5), 3e = (1,3,2)(4,6,5), 2a = (1,4)(2,5)(3,6), 6a = (1,4,2,5,3,6), 6b = (1,4,3,6,2,5).



TABLE A.20. Character table of  $C_{S_6}((1, 2, 3, 4)) \cong \langle (1, 2, 3, 4), (5, 6) \rangle \cong C_4 \times C_2$ 

	1a	2a	4a	4b	2b	2c	4c	4d
$\eta_{8.1}$	1	1	1	1	1	1	1	1
$\eta_{8.2}$	1	-1	-1	1	1	-1	-1	1
$\eta_{8.3}$	1	-1	1	-1	1	-1	1	-1
$\eta_{8.4}$	1	1	-1	-1	1	1	-1	-1
$\eta_{8.5}$	1	-1	$-E(4)$	$E(4)$	-1	1	$E(4)$	$-E(4)$
$\eta_{8.6}$	1	-1	$E(4)$	$-E(4)$	-1	1	$-E(4)$	$E(4)$
$\eta_{8.7}$	1	1	$-E(4)$	$-E(4)$	-1	-1	$E(4)$	$E(4)$
$\eta_{8.8}$	1	1	$E(4)$	$E(4)$	-1	-1	$-E(4)$	$-E(4)$

Here 1a = (), 2a = (5,6), 4a = (1,2,3,4), 4b = (1,2,3,4)(5,6), 2b = (1,3)(2,4), 2c = (1,3)(2,4)(5,6), 4c = (1,4,3,2), 4d = (1,4,3,2)(5,6).

The character table of  $C_{S_6}((1, 2, 3, 4)(5, 6)) \cong \langle (1, 2, 3, 4), (5, 6) \rangle \cong C_4 \times C_2$  is identical to the table given just above, with the only exception being the characters are indexed by 9.j. Each of the 8 irreducible characters  $\eta_{9,j}$  has the same character values as the corresponding irreducible characters  $\eta_{8,j}$  listed in the table immediately above.

TABLE A.21. Character table of  $C_{S_6}((1, 2, 3, 4, 5)) \cong \langle (1, 2, 3, 4, 5) \rangle \cong C_5$ 

	1a	5a	5b	5c	5d
$\eta_{10.1}$	1	1	1	1	1
$\eta_{10.2}$	1	$E(5)$	$E(5)^2$	$E(5)^3$	$E(5)^4$
$\eta_{10.3}$	1	$E(5)^2$	$E(5)^4$	$E(5)$	$E(5)^3$
$\eta_{10.4}$	1	$E(5)^3$	$E(5)$	$E(5)^4$	$E(5)^2$
$\eta_{10.5}$	1	$E(5)^4$	$E(5)^3$	$E(5)^2$	$E(5)$

Here 1a = (), 5a = (1,2,3,4,5), 5b = (1,3,5,2,4), 5c = (1,4,2,5,3), 5d = (1,5,4,3,2).

TABLE A.22. Character table of  $C_{S_6}((1, 2, 3, 4, 5, 6)) \cong \langle (1, 2, 3, 4, 5, 6) \rangle \cong C_6$ 

	1a	6a	3a	2a	3b	6b
$\eta_{11.1}$	1	1	1	1	1	1
$\eta_{11.2}$	1	-1	1	-1	1	-1
$\eta_{11.3}$	1	$E(3)^2$	$E(3)$	1	$E(3)^2$	$E(3)$
$\eta_{11.4}$	1	$-E(3)^2$	$E(3)$	-1	$E(3)^2$	$-E(3)$
$\eta_{11.5}$	1	$E(3)$	$E(3)^2$	1	$E(3)$	$E(3)^2$
$\eta_{11.6}$	1	$-E(3)$	$E(3)^2$	-1	$E(3)$	$-E(3)^2$

Here 1a = (), 6a = (1,2,3,4,5,6), 3a = (1,3,5)(2,4,6), 2a = (1,4)(2,5)(3,6), 3b = (1,5,3)(2,6,4), 6b = (1,6,5,4,3,2).

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## REFERENCES

- [B1] P. Bantay, The Frobenius-Schur indicator in conformal field theory, *Physics Lett. B* 394 (1997), no. 1-2, 87-88.
- [B2] P. Bantay, Frobenius-Schur indicators, the Klein-bottle amplitude, and the principle of orbifold covariance, *Phys. Lett. B* 488 (2000), 207-210.
- [FGSV] J. Fuchs, A. Ch. Ganchev, K. Szlachányi and P. Vecsernyés,  $S_4$  symmetry of  $6j$  symbols and Frobenius-Schur indicators in rigid monoidal  $C^*$  categories. *J. Math. Phys.* 40 (1999), no. 1, 408-426.
- [GAP] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008, (`\protect\vrule width0pt\protect\href{http://www.gap-system.org}{http://www.gap-system.org}`).
- [IMM] M. Iovanov, G. Mason, S. Montgomery, in preparation.
- [K] Y. Kashina, On semisimple Hopf algebras of dimensions  $2^m$ , *Algebras and Representation Theory* 6 (2003), no. 4, 393-425.
- [Ke] M. Keilberg, Higher indicators for some groups and their doubles, `arXiv:1004.1763v1 [math.RT]`.
- [KMM] Y. Kashina, G. Mason and S. Montgomery, Computing the Frobenius-Schur indicator for abelian extensions of Hopf algebras, *J. Algebra* 251 (2002), 888-913.
- [KSZ1] Y. Kashina, Y. Sommerhäuser, and Y. Zhu, Self-dual modules of semisimple Hopf algebras, *J. Algebra* 257 (2002), 88-96.
- [KSZ2] Y. Kashina, Y. Sommerhäuser, and Y. Zhu, On higher Frobenius-Schur indicators, *AMS Memoirs* 181 (2006), no. 855.
- [LM] V. Linchenko and S. Montgomery, A Frobenius-Schur theorem for Hopf algebras, *Algebras and Representation Theory*, 3 (2000), 347-355.
- [MaN] G. Mason and S-H. Ng, Central invariants and Frobenius-Schur indicators for semisimple quasi-Hopf algebras, *Advances in Math* 190 (2005), 161-195.
- [Mo] S. Montgomery, *Hopf Algebras and their Actions of Rings*, CBMS Lectures, Vol. 82, AMS, Providence, RI, 1993.
- [N1] S. Natale, On group-theoretical Hopf algebras and exact factorizations of finite groups, *J. Algebra* 270 (2003), 199-211.
- [N2] S. Natale, Frobenius-Schur indicators for a class of fusion categories, *Pacific J. Math* 221 (2005), 363-377.
- [NS1] S-H. Ng and P. Schauenburg, Central invariants and higher indicators for semisimple Quasi-Hopf algebras, *Trans. Amer. Math. Soc.* 360 (2008), 1839-1860.
- [NS2] S-H. Ng and P. Schauenburg, Frobenius-Schur indicators and exponents of spherical categories, *Advances in Math* 211 (2007) no. 1, 34-71.
- [NS3] S-H. Ng and P. Schauenburg, Higher Frobenius-Schur indicators for pivotal categories. *Hopf algebras and generalizations*, 63-90, *Contemp. Math.*, 441, *Amer. Math. Soc.*, Providence, RI, 2007.
- [R] J. J. Rotman, *An Introduction to the Theory of Groups*, Springer-Verlag, Berlin and New York, 1995.
- [S] T. Scharf, Die Wurzelanzahlfunktion in symmetrischen Gruppen, *J. Algebra* 139 (1991), pp. 446-457.
- [Se] J. P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, Berlin and New York, 1977.

UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089-1113

*Current address:* Pasadena City College, Pasadena, CA 91106

*E-mail address:* rebecca@usc.edu, recourter@pasadena.edu